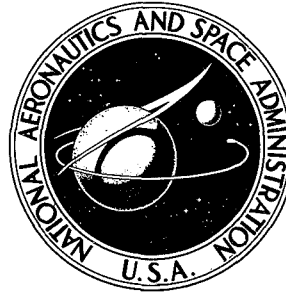


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NONLINEAR FLUID OSCILLATIONS IN
A PARTIALLY FILLED AXISYMMETRIC
CONTAINER OF GENERAL SHAPE

by John Ray Admire

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| 16. Abstract A solution for finite-amplitude free oscillations of fluid in a partially filled axisymmetric container is presented. The formulation of the problem resulted in a nonlinear boundary value problem where the nonlinearity occurred in the boundary condition at the free surface of the fluid. The boundary condition at the container wall and the differential equation are linear. The solution is obtained by first linearizing the free surface boundary condition and solving the resulting linear boundary value problem. Then, using the linear solutions, which satisfy the differential equation and the boundary condition at the container wall, a solution is found that satisfies the nonlinear boundary condition at the free surface asymptotically. A solution through the third-order term is developed. Numerical results are obtained for finite-amplitude standing waves near the first antisymmetric linear mode in three axisymmetric vessels. Results are found for three containers, a cylindrical tank with an ellipsoidal bulkhead, a cylindrical tank with a conical bulkhead, and a cylindrical tank with a truncated conical bulkhead. | | | |
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NONLINEAR FLUID OSCILLATIONS IN A PARTIALLY FILLED AXISYMMETRIC CONTAINER OF GENERAL SHAPE

INTRODUCTION

The motion of a fluid, having a free surface, has been the subject of much interest to men of inquiring minds throughout history. Problems such as the wave motion over sloping beaches, flood waves in the rivers, and the motion of ships in the sea preoccupied scientists in earlier days. Lagrange investigated surface wave problems and was a forerunner in this field to such noted scientists as Cauchy, Poisson, Airy, Stokes, Kelvin, Rayleigh, Lamb, and Stoker. These men made a great number and variety of studies of the physical problems involving surface waves and were outstanding in their contributions to this field.

Research in the field of surface wave problems, in the last two decades, has produced more information than all preceding studies. One reason for the rapid pace of surface wave research in this period can be attributed to the advent of high speed computers which permitted the use of methods of analysis that were not practical earlier. A second reason is that the aerospace industry needed vast amounts of surface wave information to design large jet aircraft and space vehicles.

This investigation is concerned with the type of surface wave problems known as standing gravity waves of finite-amplitude. W. G. Penny and A. I. Price [1] analyzed such waves in a rectangular tank of infinite depth and published the first study of finite-amplitude standing waves in 1952. Since then, a number of publications have appeared generalizing and extending their work.

Tadjbakhsh and Keller [2] analyzed finite-amplitude standing waves in a rectangular tank, as did Penny and Price; however, their analysis was for a finite depth tank. They found that finite standing gravity waves in a rectangular container will have a lower frequency than infinitesimal standing waves at large depths, but they have higher frequency below a certain depth.

Fultz [3], disturbed by the frequency effect reversal found by Tadjbakhsh and Keller, undertook an experimental study of finite-amplitude standing gravity waves in rectangular tanks. He found experimentally, as did Tadjbakhsh and Keller theoretically, that there is a frequency effect reversal of a fluid in a rectangular tank.

Mack [4] analyzed standing gravity waves of finite-amplitude in a circular cylinder and obtained results for the first axisymmetric standing wave. DiMaggio and Rehm [5] then studied the same circular cylinder to obtain the first nonaxisymmetric standing wave.

Baird [6] investigated the first nonaxisymmetric standing wave in a cylindrical sector container.

Moiseyev [7] published a general theory for free and forced oscillations of a fluid in a rigid container. To the author's knowledge, this method has not been used to obtain finite-amplitude gravity waves because of the inherent difficulty of solving the integral equations upon which the theory is based.

The problem set forth here is to obtain the first nonaxisymmetric finite-amplitude gravity wave in an axisymmetric rigid container. It is assumed that the fluid is perfect and surface tension effects are negligible.

The formulation of the problem results in a nonlinear boundary value problem. The nonlinearity is in the boundary condition at the free surface of the fluid. Not only is the boundary condition at the free surface nonlinear, but also it must be satisfied on a moving surface of unknown shape. The method of solution is to expand the nonlinear boundary in a Taylor's series about the undisturbed free surface, and then satisfy it asymptotically by the method of Krylov and Bogoliubov [8].

The solution to this problem is of dual interest. Not only is it a fundamental study in the field of surface waves, but also it provides a means to obtain a thorough understanding of surface waves in space vehicle propellant tanks. In no other application to axisymmetric containers is the motion of the fluid more important than in the field of space research since the

propellant in a space vehicle contributes over 90 percent of the total weight.

Thus, this study contributes much information about the overall dynamic characteristics of a space vehicle.

GOVERNING EQUATIONS FOR FREE OSCILLATIONS OF A FLUID IN A RIGID, PARTIALLY FILLED AXISYMMETRIC CONTAINER

The volume of fluid, V , in the axisymmetric container is bounded by the free surface, S , and the surface in contact with the container, F . The free surface existing when the fluid is at rest is known as the undisturbed free surface, S_0 .

In this analysis the fluid is assumed to be incompressible and inviscid with its motion irrotational. The effects of surface tension are neglected. The only body force considered to be acting on the fluid is that due to gravity, and it acts along the negative z -axis as shown in Figure 1. These assumptions are justifiable [9] in the study of many fluid oscillation problems, such as the sloshing of the propellant in the tanks of a space vehicle during boost flight.

The above assumptions ensure the existence of a velocity potential $\psi(r, \theta, z; t)$ for the fluid [10]. Thus, the velocity field can be expressed as

$$\vec{v} = \nabla \psi \quad \text{in } V, \quad (2.1)$$

and the continuity equation, in terms of the velocity potential, becomes

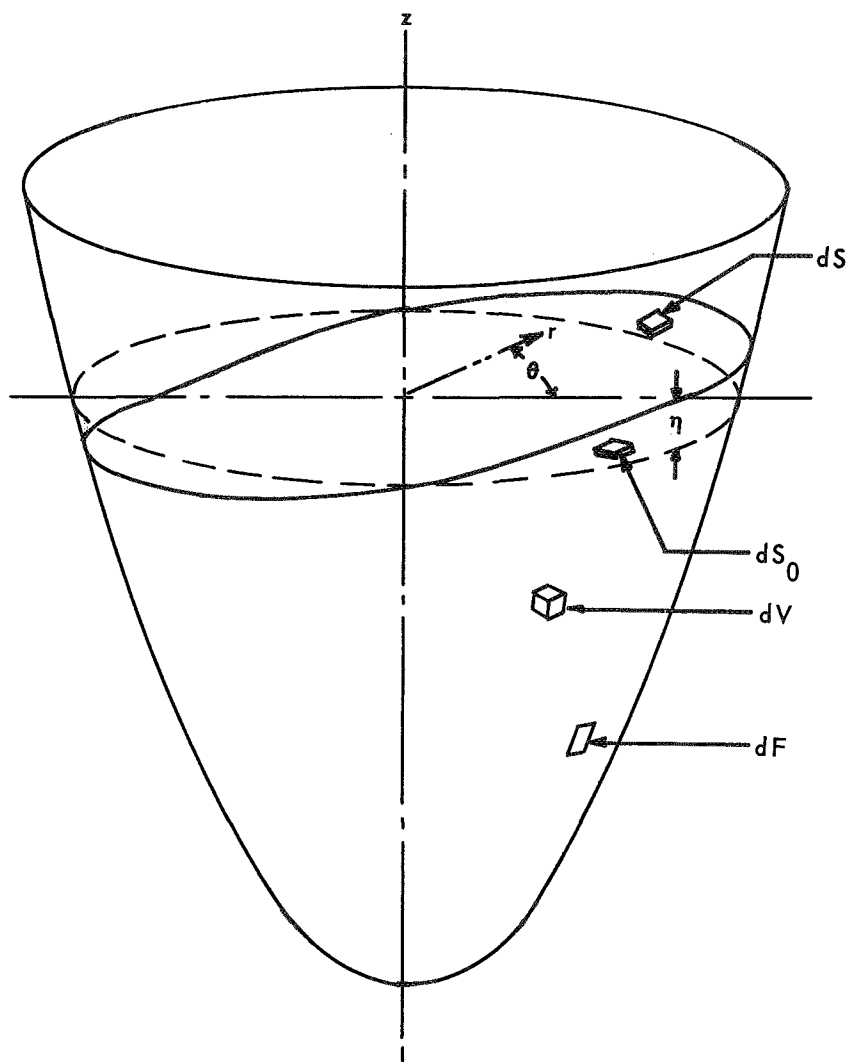


Figure 1. Coordinate System

Laplace's equation

$$\nabla^2 \psi = 0 \quad \text{in } V \quad (2.2)$$

Thus, the equation of motion of the fluid is Laplace's equation, a linear differential equation about which a great deal is known.

The boundary conditions that the solutions of (2.2) must satisfy are based on the assumptions that any particle on the boundary of the fluid remains on the boundary and that the pressure on the free surface is known. To determine mathematical expressions for the boundary conditions, let the entire surface of the fluid, both fixed and free, be expressed as

$$\delta(r, \theta, z; t) = 0 \quad (2.3)$$

Taking the material derivative of (2.3) yields

$$\nabla \delta \cdot \nabla \psi + \delta_t = 0 \quad (2.4)$$

Noting that $\nabla \delta \cdot \nabla \psi$ is the velocity normal to the surface S , equation (2.4) becomes

$$\partial \psi / \partial \mathbf{n} + \delta_t = 0 \quad \text{on } S \quad (2.5)$$

Since $\delta_t = 0$ on the fluid surface in contact with the container, the boundary condition on F becomes

$$\partial \psi / \partial \mathbf{n} = 0 \quad \text{on } F \quad (2.6)$$

Letting $\eta(r, \theta; t)$ be the wave height of the free surface, δ may be expressed as

$$\delta = z - \eta \quad \text{on } S \quad . \quad (2.7)$$

Then substituting (2.7) in (2.5), the boundary condition on the free surface becomes

$$\psi_r \eta_r + \psi_\theta \eta_\theta / r^2 - \psi_z + \eta_t = 0 \quad \text{on } S \quad . \quad (2.8)$$

It is necessary to obtain another boundary condition for the free surface since the unknown wave height η has been introduced into the problem. The additional boundary condition comes from the previously mentioned assumption that the pressure on the free surface is known. A mathematical expression for this boundary comes from Bernoulli's law

$$\psi_t + (\nabla \psi \cdot \nabla \psi) / 2 + (P - P_0) / \rho + gz = 0 \quad , \quad (2.9)$$

where P_0 is the pressure at the free surface. On the free surface S , equation (2.9) becomes

$$\psi_t + (\nabla \psi \cdot \nabla \psi) / 2 + \eta g = 0 \quad \text{on } S \quad (2.10)$$

The differential equation and appropriate boundary conditions that govern the motion of a fluid undergoing free oscillations in a rigid, partially filled container have been developed. These governing equations are summarized as follows:

$$\nabla^2 \psi = 0 \quad \text{in } V \quad (2.11a)$$

$$\partial \psi / \partial n = 0 \quad \text{on } F \quad (2.11b)$$

$$\psi_r \eta_r + \psi_\theta \eta_\theta / r^2 - \psi_z + \eta_t = 0 \quad \text{on } S \quad (2.11c)$$

$$\psi_t + (\nabla \psi \cdot \nabla \psi) / 2 + \eta g = 0 \quad \text{on } S \quad . \quad (2.11d)$$

Equations (2.11) represent a boundary value problem in which the differential equation is linear, and the boundary conditions consist of one linear boundary condition and two nonlinear boundary conditions. Not only are equations (2.11c) and (2.11d) nonlinear, but they must be satisfied on a moving surface. Because of the difficulty of solving such a boundary value problem, very little progress has been made without making simplifying assumptions of one sort or another.

The assumption is made in this analysis that the wave height η and the velocity potential ψ can be expanded in a power series with respect to a small positive parameter ϵ , which is proportional to the wave height of linear solution and becomes evident later in the analysis.

$$\psi = \epsilon \psi^1 + \epsilon^2 \psi^2 + \epsilon^3 \psi^3 + \dots \quad (2.12)$$

$$\eta = \epsilon \eta^1 + \epsilon^2 \eta^2 + \epsilon^3 \eta^3 + \dots \quad (2.13)$$

Equations (2.11a) and (2.11b), through the use of (2.12), become

$$\nabla^2 \psi^{(i)} = 0 \quad \text{in } V \quad (2.14)$$

$$\partial \psi^{(i)} / \partial n = 0 \quad \text{on } F \quad (2.15)$$

Since boundary conditions (2.11c) and (2.11d) require evaluating the velocity potential and its derivatives on the free surface ($z = \eta$), it is assumed that these functions can be expanded in a Taylor's series about the undisturbed free surface (that is, $z = 0$). Thus

$$\begin{aligned}\psi(r, \theta, \eta; t) &= \psi(r, \theta, 0; t) + \psi_z(r, \theta, 0; t)\eta \\ &+ \psi_{zz}(r, \theta, 0; t)\eta^2/2 + \dots\end{aligned}\quad (2.16)$$

Substituting equations (2.12) and (2.13) in (2.16), yields

$$\begin{aligned}\psi(r, \theta, \eta; t) &= \epsilon[\psi^1(r, \theta, 0; t)] + \epsilon^2[\psi^2(r, \theta, 0; t) + \psi_z^1(r, \theta, 0; t)\eta^1] \\ &+ \epsilon^3[\psi^3(r, \theta, 0; t) + \psi_z^2(r, \theta, 0; t)\eta^1 + \psi_z^1(r, \theta, 0; t)\eta^2 \\ &+ \psi_{zz}^1(r, \theta, 0; t)\eta^1\eta^1] + O(\epsilon^4)\end{aligned}\quad (2.17)$$

Since the derivatives of ψ have similar expansions, equation (2.11c) becomes

$$\begin{aligned}&\epsilon(\eta_t^1 - \psi_z^1) + \epsilon^2(\eta_t^2 - \psi_z^2 - \psi_{zz}^1\eta^1 + \psi_r^1\eta_r^1 + \psi_\theta^1\eta_\theta^1/r^2) \\ &+ \epsilon^3(\eta_t^3 - \psi_t^3 - \psi_{zz}^1\eta^2 - \psi_{zz}^2\eta^1 - \psi_{zzz}^1\eta^1\eta^1 + \eta_r^2\psi_r^1 + \eta_r^1\psi_r^2 \\ &+ \eta_r^1\psi_{rz}^1\eta^1 + \eta_\theta^2\psi_\theta^1/r^2 + \eta_\theta^1\psi_\theta^2/r^2 + \eta_\theta^1\psi_\theta^1\eta^1/r^2) \\ &+ O(\epsilon^4) = 0 \quad \text{on } S_0, \end{aligned}\quad (2.18)$$

and equation (2.11d) becomes

$$\begin{aligned}
& \epsilon(\psi_t^1 + \eta^1 g) + \epsilon^2\{\psi_t^2 + \psi_{tz}^1 \eta^1 + g\eta^2 \\
& \quad + [(\psi_r^1)^2 + (\psi_\theta^1)^2/r^2 + (\psi_z^1)^2]/2\} \\
& + \epsilon^3(\psi_t^3 + \psi_{tz}^1 \eta^2 + \psi_{tz}^2 \eta^1 + \psi_{zzt}^1 \eta^1 \eta^1 + g\eta^3 + \psi_r^1 \psi_r^2 + \psi_r^1 \psi_{rz}^1 \eta^1 \\
& \quad + \psi_\theta^1 \psi_\theta^2/r^2 + \psi_\theta^1 \psi_{\theta z}^1 \eta^1/r^2 + \psi_z^1 \psi_z^2 + \psi_z^1 \psi_{zz}^1 \eta^1) \\
& + O(\epsilon^4) = 0 \quad \text{on } S_0 \quad (2.19)
\end{aligned}$$

Equating the coefficients of ϵ^n equal to zero, the equations to be solved for the first-order become

$$\nabla^2 \psi^1 = 0 \quad \text{in } V \quad (2.20a)$$

$$\partial \psi^1 / \partial n = 0 \quad \text{on } F \quad (2.20b)$$

$$\eta_t^1 - \psi_z^1 = 0 \quad \text{on } S_0 \quad (2.20c)$$

$$\psi_t^1 + g\eta^1 = 0 \quad \text{on } S_0 \quad (2.20d)$$

For the second-order, the equations become

$$\nabla^2 \psi^2 = 0 \quad \text{in } V \quad (2.21a)$$

$$\partial \psi^2 / \partial n = 0 \quad \text{on } F \quad (2.21b)$$

$$\eta_t^2 - \psi_t^2 = \psi_{zz}^1 \eta^1 - \psi_r^1 \eta_r^1 - \psi_\theta^1 \eta_\theta^1 / r^2 \quad \text{on } S_0 \quad (2.21c)$$

$$\begin{aligned}
\psi_t^2 + g\eta^2 = & -\psi_{tz}^1 \eta^1 - [(\psi_r^1)^2 + (\psi_\theta^1)^2/r^2 \\
& + (\psi_z^1)^2] \quad \text{on } S_0 \quad (2.21d)
\end{aligned}$$

And, for the third-order, the equations become

$$\nabla^2 \psi^3 = 0 \quad \text{in } V \quad (2.22a)$$

$$\partial \psi^3 / \partial n = 0 \quad \text{on } F \quad (2.22b)$$

$$\begin{aligned} \eta_t^3 - \psi_t^3 = & \psi_{zz}^1 \eta^2 + \psi_{zz}^2 \eta^1 + \psi_{zzz}^1 \eta^1 \eta^1 - \eta_r^2 \psi_r^1 - \eta_r^1 \psi_r^2 \\ & - \eta_r^1 \eta^1 \psi_{rz}^1 - \eta_\theta^2 \psi_\theta^1 / r^2 - \eta_\theta^1 \psi_\theta^2 / r^2 - \eta_\theta^1 \psi_{\theta z}^1 \eta^1 / r^2 \\ & \text{on } S_0 \end{aligned} \quad (2.22c)$$

$$\begin{aligned} \psi_t^3 + g\eta^3 = & -\psi_{tz}^1 \eta^2 - \psi_{tz}^2 \eta^1 - \psi_{zzt}^1 \eta^1 \eta^1 - \psi_r^1 \psi_r^2 - \psi_r^1 \psi_{rz}^1 \eta^1 \\ & - \psi_\theta^1 \psi_\theta^2 / r^2 - \psi_\theta^1 \psi_{\theta z}^1 \eta^1 / r^2 - \psi_z^1 \psi_z^2 - \psi_z^1 \psi_{zz}^1 \eta^1 \\ & \text{on } S_0 \end{aligned} \quad (2.22d)$$

Equations (2.20) through (2.22), in principle, furnish a means of calculating successively the coefficients of the series (2.12) and (2.13). Equations (2.20) lead to solutions for ψ^1 and η^1 . Once ψ^1 and η^1 are determined, they can be inserted in equations (2.21) to permit solutions for ψ^2 and η^2 . Thus, this procedure can be applied until as many terms in the series for ψ and η as desired have been calculated.

Even though the procedure described above will yield a solution, there remain two boundary conditions on the free surface and two unknowns in the problem to determine. Therefore, it is a computational advantage to combine the two free surface boundary conditions so that η is eliminated, leaving a boundary value problem with one unknown to determine. This is simple to do once equations (2.20d), (2.21d), and (2.22d) have been differentiated with

respect to time. It must be remembered that combining the two free surface boundary equations in this manner eliminates any constant term which may exist in η^1 , η^2 , and η^3 . These constant terms will be dealt with later. The boundary value problem to be solved, after the algebraic manipulations have been performed, is for the first-order,

$$\nabla^2 \psi^1 = 0 \quad \text{in } V \quad (2.23a)$$

$$\partial \psi^1 / \partial n = 0 \quad \text{on } F \quad (2.23b)$$

$$\psi_{tt}^1 + g \psi_z^1 = 0 \quad \text{on } S_0 \quad (2.23c)$$

and

$$\eta^1 = -\psi_t^1 / g \quad \text{on } S_0 \quad (2.23d)$$

For the second-order, the problem is

$$\nabla^2 \psi^2 = 0 \quad \text{in } V \quad (2.24a)$$

$$\partial \psi^2 / \partial n = 0 \quad \text{on } F \quad (2.24b)$$

$$\begin{aligned} \psi_{tt}^2 + g \psi_z^2 = & \psi_t^1 \psi_{ttz}^1 / g + \psi_{zz}^1 \psi_t^1 - 2\psi_r^1 \psi_{rt}^1 - 2\psi_\theta^1 \psi_{\theta t}^1 / r^2 \\ & - 2\psi_z^1 \psi_{zt}^1 \quad \text{on } S_0 \end{aligned} \quad (2.24c)$$

and

$$\begin{aligned} \eta^2 = & -\psi_t^2 / g - \psi_{tz}^1 \psi_t^1 / g^2 - [(\psi_r^1)^2 + (\psi_\theta^1)^2 / r^2 + (\psi_z^1)^2] / (2g) \\ & \text{on } S_0 \end{aligned} \quad (2.24d)$$

And, for the third-order, the problem is

$$\nabla^2 \psi^3 = 0 \quad \text{in } V \quad (2.25a)$$

$$\partial \psi^3 / \partial \mathbf{n} = 0 \quad \text{on } F \quad (2.25b)$$

$$\begin{aligned} \psi_{tt}^3 + g\psi_z^3 = & -2\psi_r^1\psi_{rt}^2 - 2\psi_r^2\psi_{rt}^1 - 2\psi_\theta^1\psi_{\theta t}^2/r^2 - 2\psi_\theta^2\psi_{\theta t}^1/r^2 \\ & - 2\psi_z^1\psi_{zt}^2 - 2\psi_z^2\psi_{zt}^1 + \psi_{ztt}^1\psi_t^2/g + \psi_{ztt}^2\psi_t^1/g \\ & + \psi_{zz}^1\psi_t^2 + \psi_{zz}^2\psi_t^1 - (\psi_r^1)^2\psi_{rr}^1 - (\psi_\theta^1)^2\psi_{\theta\theta}^1/r^4 \\ & - (\psi_z^1)^2\psi_{zz}^1 + (\psi_\theta^1)^2\psi_r^1/r^3 - 2\psi_r^1\psi_z^1\psi_{rz}^1 - 2\psi_r^1\psi_\theta^1\psi_{r\theta}^1/r^2 \\ & - 2\psi_z^1\psi_\theta^1\psi_{\theta z}^1/r^2 + \psi_{ttz}^1(\psi_r^1)^2/(2g) + (\psi_\theta^1)^2\psi_{ttz}^1/(2gr^2) \\ & + \psi_{ttz}^1(\psi_z^1)^2/(2g) + \psi_{zz}^1(\psi_r^1)^2/2 + \psi_{zz}^1(\psi_\theta^1)^2/(2r^2) \\ & + \psi_{zz}^1(\psi_z^1)^2/2 + 2\psi_t^1\psi_r^1\psi_{rt}^1/g + 2\psi_t^1\psi_r^1\psi_{rz}^1/g \\ & + 2\psi_t^1\psi_\theta^1\psi_{\theta t}^1/(gr^2) + 2\psi_t^1\psi_\theta^1\psi_{\theta z}^1/(gr^2) + 2\psi_t^1\psi_{zz}^1\psi_{zt}^1/g \\ & + 2\psi_t^1\psi_z^1\psi_{zzt}^1/g - \psi_{zt}^1\psi_t^1\psi_{ztt}^1/g^2 - \psi_{zt}^1\psi_t^1\psi_{zz}^1/g \\ & - (\psi_t^1)^2\psi_{ttzz}^1/(2g^2) - (\psi_t^1)^2\psi_{zzz}^1/(2g) \end{aligned}$$

on S_0 (2.25c)

and

$$\begin{aligned} \eta^3 = & -\psi_t^3/g + \psi_{tz}^1\psi_t^2/g^2 - (\psi_{tz}^1)^2\psi_t^1/g^3 \\ & + \psi_{tz}^1[(\psi_r^1)^2 + (\psi_\theta^1)^2/r^2 + (\psi_z^1)^2]/(2g^2) + \psi_{tz}^2\psi_t^1/g^2 \\ & + \psi_{zzt}^1(\psi_t^1)^2/g^2 + \psi_r^1\psi_r^2 + \psi_r^1\psi_r^1\psi_t^1/g^2 \quad \text{on } S_0 \end{aligned} \quad (2.25d)$$

The constant terms that were removed from η^1 , η^2 , and η^2 in the process of combining the free surface boundary equations may be lumped together in one constant, η^0 . The wave height equation then becomes

$$\eta = \eta^0 + \epsilon \eta^1 + \epsilon^2 \eta^2 + \epsilon^3 \eta^3 + \dots \quad \text{on } S_0 \quad . \quad (2.26)$$

The constant η^0 is evaluated from the condition that the volume of fluid remains constant and can be expressed mathematically as

$$\int_{S_0} \eta dS_0 = 0 \quad . \quad (2.27)$$

The equations governing the nonlinear oscillations of a fluid in a partially filled container have been developed.

LINEAR SOLUTION

It is common in many fluid oscillation problems to retain only the first-order terms in the governing equations. The problem is then known as a linear fluid oscillation problem. This linearization is accomplished by truncating the series expansions of ψ and η after the first term; therefore, the governing equations are the first-order equations expressed by (2.23). These equations are restated as

$$\nabla^2 \psi^1 = 0 \quad \text{in } V \quad (3.1a)$$

$$\partial \psi^1 / \partial n = 0 \quad \text{on } F \quad (3.1b)$$

$$\psi_{tt}^1 + g \psi_z^1 = 0 \quad \text{on } S_0 \quad (3.1c)$$

and

$$\eta^1 = - \psi_t^1 / g \quad \text{on } S_0 \quad (3.1d)$$

Since this analysis is concerned only with fluid motion that is harmonic with time, the time dependence can be removed from the equations by the transformation

$$\psi^1 = \phi(r, \theta, z) e^{i\omega t}, \quad (3.2)$$

where ω is the frequency of oscillation. Then equations (3.1) become

$$\nabla^2 \phi = 0 \quad \text{in } V \quad (3.3a)$$

$$\partial \phi / \partial n = 0 \quad \text{on } F \quad (3.3b)$$

$$\phi_z = \omega^2 \phi / g \quad \text{on } S_0 \quad (3.3c)$$

The boundary value problem (3.3) can also be expressed as an extremum problem. According to Hamilton's principle

$$\delta I = 0 \quad , \quad (3.4)$$

where δI is an isochronous variation of

$$I = \int_t (\text{KE} - \text{PE}) dt \quad . \quad (3.5)$$

The kinetic and potential energy are expressed as

$$\text{KE} = (\rho/2) \int_V \vec{v} \cdot \vec{v} dV \quad (3.6)$$

$$\text{PE} = (\rho g/2) \int_{S_0} (\eta^1)^2 dS_0 \quad , \quad (3.7)$$

where ρ is the fluid density.

If the kinetic and potential energy are written in terms of the velocity potential, equation (3.5) becomes

$$I = \int_t [\rho/2 \int_V (\nabla \psi^1 \cdot \nabla \psi^1) dV - (\rho/2g) \int_{S_0} (\psi_t^1)^2 dS_0] dt \quad . \quad (3.8)$$

By integrating over t from 0 to $2\pi/\omega$ and omitting a nonessential factor, I can be expressed as

$$I = \int_V (\nabla \phi \cdot \nabla \phi) dV - (\omega^2/g) \int_{S_0} (\phi)^2 dS_0 \quad . \quad (3.9)$$

Lawrence, Wang, and Reddy [11] have shown that the solutions of equations (3.3) are also the solutions of the extremum problem for the integral I as expressed by equation (3.9).

The approach taken in this analysis is to determine an approximate solution to the extremum problem rather than to solve the boundary value equations. Trefftz's method [12] will be used to construct this approximate solution. This method begins by choosing a complete sequence of linearly independent functions, $u_{ik}(r, \theta, z)$, each of which satisfies the differential equation. The approximate solution is expressed as a series of the coordinate functions u_{ik}

$$\hat{\phi} = \sum_i \sum_k a_{ik} u_{ik} \quad , \quad (3.10)$$

where the coefficients a_{ik} are determined from the condition

$$I(\phi - \hat{\phi}) = \text{minimum} \quad . \quad (3.11)$$

A set of necessary conditions are

$$\partial I(\phi - \hat{\phi}) / \partial a_{ik} = 0 \quad . \quad (3.12)$$

Through the use of equations (3.9) and (3.10), equation (3.12)

becomes

$$\sum_j \sum_l \left\{ \int_V (\nabla u_{jl} \cdot \nabla u_{ik}) dV - (\omega^2/g) \int_{S_0} u_{jl} u_{ik} dS_0 \right\} a_{jl} = 0 \quad (3.13)$$

The volume integral in (3.13) can be changed to a surface integral using Green's Theorem. Applying this theorem to the volume integral and noting that u_{ik} and u_{jl} satisfy Laplace's equation and that the dot product obeys the commutative law, the volume integral becomes

$$\begin{aligned} \int_V (\nabla u_{jl} \cdot \nabla u_{ik}) dV &= \frac{1}{2} \int_F (u_{jl} \partial u_{ik} / \partial n + u_{ik} \partial u_{jl} / \partial n) dF \\ &+ \frac{1}{2} \int_{S_0} (u_{jl} \partial u_{ik} / \partial n + u_{ik} \partial u_{jl} / \partial n) dS_0 \quad (3.14) \end{aligned}$$

Equation (3.13) can now be written as

$$\begin{aligned} \sum_j \sum_l \left\{ \int_F (u_{jl} \partial u_{ik} / \partial n + u_{ik} \partial u_{jl} / \partial n) dF \right. \\ \left. + \int_{S_0} (u_{jl} \partial u_{ik} / \partial n + u_{ik} \partial u_{jl} / \partial n) dS_0 \right. \\ \left. - (2\omega^2/g) \int_{S_0} (u_{ik} u_{jl}) dS_0 \right\} a_{jl} = 0 \quad (3.15) \end{aligned}$$

Since this analysis is concerned with axisymmetric containers, it is advantageous to consider the surface F to be generated by rotating a curve γ about the axis of symmetry.

Let γ be expressed in parametric form

$$\mathbf{r} = \mathbf{r}(\alpha)$$

$$z = z(\alpha)$$

where

$$\alpha_1 \leq \alpha \leq \alpha_2$$

and

$$\mathbf{r}(\alpha_1) = 0 \quad z(\alpha_1) = -H$$

$$\mathbf{r}(\alpha_2) = R \quad z(\alpha_2) = 0$$

where R is the radius of the container at the free surface, and H is the height of the container at the axis of symmetry.

The derivatives normal to \mathbf{F} and differential element of area on \mathbf{F} then become

$$\partial(\)/\partial n = [z'\partial(\)/\partial \mathbf{r} - \mathbf{r}'\partial(\)/\partial z]/\sqrt{(\mathbf{r}')^2 + (z')^2} \quad (3.16)$$

$$d\mathbf{F} = r\sqrt{(\mathbf{r}')^2 + (z')^2} d\theta d\alpha \quad (3.17)$$

where

$$\mathbf{r}' = d\mathbf{r}(\alpha)/d\alpha \quad z' = dz(\alpha)/d\alpha$$

Using equations (3.16) and (3.17) and realizing that the surface S_0 is a circle, we can express equation (3.15) as

$$\begin{aligned}
& \sum_j \sum_l \left\{ \int_0^{2\pi} \int_{\alpha_1}^{\alpha_2} [u_{jl} (z' \partial u_{ik} / \partial r - r' \partial u_{ik} / \partial z) \right. \\
& \quad \left. + u_{ik} (z' \partial u_{jl} / \partial r - r' \partial u_{jl} / \partial z)] r d\alpha d\theta \right. \\
& \quad + \int_0^{2\pi} \int_0^R [u_{jl} \partial u_{ik} / \partial z + u_{ik} \partial u_{jl} / \partial z] r dr d\theta \\
& \quad \left. - (2\omega^2/g) \int_0^{2\pi} \int_0^R (u_{ik} u_{jl}) r dr d\theta \right\} a_{jl} = 0 \quad (3.18)
\end{aligned}$$

The chief difficulty encountered in solving equations (3.18) is selecting the coordinate functions since there are no general recommendations available. It is known from the general theory that if the system of coordinate functions is complete and it satisfies the differential equation, the minimizing sequence constructed by Trefftz's method will converge to the exact solution [12]. It is therefore advisable to select u_{ik} as eigenfunctions of some volume that contains the given volume but has a simpler shape. Therefore, in this analysis, the coordinate functions that will be used are the eigenfunctions of the fluid in a cylindrical container whose cross section equals the largest cross section of the given container. Thus

$$u_{ik} = J_k(\beta_{ki} r) \cosh[\beta_{ki} (z + H_c)] \cos(k\theta) / \cosh(\beta_{ki} H_c) \quad (3.19)$$

where β_{ki} are the roots of $dJ_k(\beta r)/dr \big|_{r=R_c} = 0$; R_c is the maximum radius of the axisymmetric container; and H_c is the depth of the fluid in the axisymmetric container.

Equation (3.19) can be rewritten as

$$u_{ik} = \hat{u}_{ik} \cos(k\theta) \quad , \quad (3.20)$$

where

$$\hat{u}_{ik} = J_k(\beta_{ki}r) \cosh[\beta_{ki}(z + H_c)] / \cosh(\beta_{ki}H_c)$$

Inserting the coordinate functions expressed by (3.20) into (3.18)

and integrating with respect to θ , yields

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \left\{ \int_{\alpha_1}^{\alpha_2} [\hat{u}_{jk} (z' \partial \hat{u}_{ik} / \partial r - r' \partial \hat{u}_{ik} / \partial z) \right. \\ \left. + \hat{u}_{ik} (z' \partial \hat{u}_{jk} / \partial r - r' \partial \hat{u}_{jk} / \partial z)] r d\alpha \right. \\ \left. + \int_0^R (\hat{u}_{jk} \partial \hat{u}_{ik} / \partial z + \hat{u}_{ik} \partial \hat{u}_{jk} / \partial z) r dr \right. \\ \left. - (2\omega^2/g) \int_0^R (\hat{u}_{jk} \hat{u}_{ik}) r dr \right\} a_{jk} = 0 \quad , \quad (3.21) \end{aligned}$$

where a nonessential factor has been dropped and the orthogonal properties of the cosine functions have been used.

Equation (3.21) represents an eigenvalue problem for each k .

Since equation (3.21) is symmetric with respect to i and j , the eigenvalues, $2\omega^2/g$, are real. Solving equation (3.21) by one of several standard means for each value of k , the eigenvalues and vectors are

$2\lambda_{kl}/g$ — eigenvalue

a_{jkl} — j th element of the eigenvector .

Thus the solutions of (3.3) are

$$\phi_{kl} = \sum_{j=1}^{\infty} a_{jkl} J_k(\beta_{kj} r) \cosh[\beta_{kj}(z + H_c)] \cos(k\theta) / \cosh(\beta_{kj} H_c) \quad (3.22)$$

and the frequency of oscillation is

$$\omega_{kl} = \sqrt{\lambda_{kl}} \quad (3.23)$$

Equations (3.22) and (3.23) are the eigenfunctions and the frequency of oscillation of a fluid in an axisymmetric container.

The eigenfunctions have the following important orthogonality relationship which will be needed later:

$$\int_{S_0} \phi_{ki} \phi_{lj} dS_0 = \delta_{kl} \delta_{ij} \quad (3.24)$$

This relationship is easily obtained by using the orthogonal properties of the cosine functions and the eigenvectors.

A means has been presented by which the linear fluid oscillation problem can be solved for the case of a fluid contained in an axisymmetric vessel of general shape.

NONLINEAR SOLUTION

The equations governing the nonlinear fluid oscillation problem were developed previously. The velocity potential and wave height were assumed to be expandable in a power series of a small positive parameter, and the governing equations were developed for the terms in the expansion for the velocity potential through the third-order term. The method by which the nonlinear equations will be solved is that of Krylov and Bogoliubov [8].

This method assumes that velocity potential can be expressed as

$$\psi^{(i)} = \sum_{p=0}^{\infty} \sum_{m=1}^{\infty} A_{pm}^{(i)} \phi_{pm} \quad (4.1)$$

It is obvious that the assumed form of $\psi^{(i)}$ satisfies the differential equation and the boundary condition on F , since ϕ_{pm} are solutions of the linear problem. Thus, the problem at hand is to evaluate $A_{pm}^{(i)}$ so that the free surface boundary condition is satisfied. Since we are seeking periodic solutions, $A_{pm}^{(i)}$ is periodic with time; therefore, it is advantageous to effect a change of variable so that $A_{pm}^{(i)}$ is periodic with a period of 2π . Thus let

$$\tau = \int_0^t (\omega + \epsilon B^1 + \epsilon^2 B^2 + \dots) dt \quad , \quad (4.2)$$

where ω is the linear frequency.

Therefore, the instantaneous frequency $\dot{\tau}$ is obtained by differentiating equation (4.2)

$$\dot{\tau} = \omega + \epsilon B^1 + \epsilon^2 B^2 + \dots \quad (4.3)$$

In general, the terms $B^{(i)}$ are functions of the wave amplitude.

Thus, for the sake of completeness, let equation (4.3) be written as

$$\dot{\tau} = \omega + \epsilon B^1(K) + \epsilon^2 B^2(K) + \dots \quad (4.4)$$

where K is an amplitude parameter, and its rate of change is a function of the amplitude. Thus K can be defined by

$$\dot{K} = \epsilon D^1(K) + \epsilon^2 D^2(K) + \dots \quad (4.5)$$

Equation (4.1) can now be written as

$$\psi^{(i)} = \sum_{p=0}^{\infty} \sum_{m=1}^{\infty} A_{pm}^{(i)}(\tau, K) \phi_{pm} \quad , \quad (4.6)$$

where the parameters τ and K are defined by

$$\dot{\tau} = \omega + \epsilon B^1(K) + \epsilon^2 B^2(K) + \dots \quad (4.7)$$

$$\dot{K} = \epsilon D^1(K) + \epsilon^2 D^2(K) + \dots \quad (4.8)$$

To uniquely define the coefficients in equations (4.7) and (4.8) it is necessary to impose additional restrictions on them. Mathematically, these conditions can be expressed as

$$\int_0^{2\pi} \int_{S_0} \psi K \cos \tau \phi_{11} dS_0 d\tau = \epsilon K^2 \pi \quad (4.9)$$

$$\int_0^{2\pi} \int_{S_0} \psi K \sin \tau \phi_{11} dS_0 d\tau = 0 \quad (4.10)$$

From a physical point of view, the imposition of these conditions is equivalent to selecting K as the full amplitude of the first harmonic of the oscillation.

Noting that $\psi = \psi[r, \theta, z, \tau(t), K(t)]$, the time derivatives of the velocity potential become

$$\psi_t = \psi_\tau \dot{\tau} + \psi_K \dot{K} \quad (4.11)$$

$$\psi_{tt} = (\psi_t)_\tau \dot{\tau} + (\psi_t)_K \dot{K} \quad (4.12)$$

Substituting the expansions for $\dot{\tau}$, \dot{K} , and ψ in equations (4.11) and (4.12) yields

$$\begin{aligned} \psi_t = & \epsilon(\omega \psi_\tau^1) + \epsilon^2(\omega \psi_\tau^2 + B^1 \psi_\tau^1 + D^1 \psi_K^1) \\ & + \epsilon^3(\omega \psi_\tau^3 + B^1 \psi_\tau^2 + B^2 \psi_\tau^1 + D^1 \psi_K^2 + D^2 \psi_K^1) + O(\epsilon^4) \end{aligned} \quad (4.13)$$

and

$$\begin{aligned}
\psi_{tt} = & \epsilon(\omega^2\psi_{\tau\tau}) + \epsilon^2(\omega^2\psi_{\tau\tau}^2 + 2\omega B^1\psi_{\tau\tau}^1 + 2\omega D^1\psi_{K\tau}^1) \\
& + \epsilon^3[\omega^2\psi_{\tau\tau}^3 + D^1B_K^1\psi_{\tau}^1 + (B^1)^2\psi_{\tau\tau}^1 + 2\omega B^2\psi_{\tau\tau}^1 \\
& + 2\omega B^1\psi_{\tau\tau}^2 + 2\omega D^1\psi_{K\tau}^2 + 2B^1D^1\psi_{K\tau}^1 + 2\omega D^2\psi_{K\tau}^1 \\
& + (D^1)^2\psi_{KK}^1 + D^1D_K^1\psi_K^1] + O(\epsilon^4) \quad , \quad (4.14)
\end{aligned}$$

Substituting the expansions for ψ_t and ψ_{tt} in equations (4.13) and (4.14) and equating coefficients of ϵ^n , yields

$$\psi_t^1 = \omega\psi_{\tau}^1 \quad (4.15)$$

$$\psi_t^2 = \omega\psi_{\tau}^2 + B^1\psi_{\tau}^1 + D^1\psi_K^1 \quad (4.16)$$

$$\psi_t^3 = \omega\psi_{\tau}^3 + B^1\psi_{\tau}^2 + B^2\psi_{\tau}^1 + D^1\psi_K^2 + D^2\psi_K^1 \quad (4.17)$$

$$\psi_{tt}^1 = \omega^2\psi_{\tau\tau}^1 \quad (4.18)$$

$$\psi_{tt}^2 = \omega^2\psi_{\tau\tau}^2 + 2\omega B^1\psi_{\tau\tau}^1 + 2\omega D^1\psi_{K\tau}^1 \quad (4.19)$$

$$\begin{aligned}
\psi_{tt}^3 = & \omega^2\psi_{\tau\tau}^3 + D^1B_K^1\psi_{\tau}^1 + 2\omega B^2\psi_{\tau\tau}^1 + (B^1)^2\psi_{\tau\tau}^1 + 2\omega B^1\psi_{\tau\tau}^2 \\
& + 2\omega D^1\psi_{K\tau}^2 + 2B^1D^1\psi_{K\tau}^1 + 2\omega D^2\psi_{K\tau}^1 + (D^1)^2\psi_{KK}^1 \\
& + D^1D_K^1\psi_K^1 \quad (4.20)
\end{aligned}$$

Through the use of equations (4.15) through (4.20), the boundary condition equations (2.23c), (2.24c), and (2.25c) become, for the first-order

$$\omega^2 \psi_{\tau\tau}^1 + g \psi_z^1 = 0, \quad ,$$

and, for the second-order

$$\begin{aligned} \omega^2 \psi_{\tau\tau}^2 + g \psi_z^2 = & -2B^1 \omega \psi_{\tau\tau}^1 - 2\omega D^1 \psi_{K\tau}^1 - 2\omega \psi_{\tau r}^1 \psi_r^1 \\ & - 2\omega \psi_{\tau\theta}^1 \psi_\theta^1 / r^2 - 2\omega \psi_{\tau z}^1 \psi_z^1 + \omega^3 \psi_{\tau\tau z}^1 \psi_\tau^1 / g \\ & + \omega \psi_\tau^1 \psi_{zz}^1, \end{aligned} \quad (4.22)$$

and, finally, for the third-order

$$\begin{aligned} \omega^2 \psi_{\tau\tau}^3 + g \psi_z^3 = & -B_K^1 D^1 \psi_\tau^1 - 2\omega B^2 \psi_{\tau\tau}^1 - (B^1)^2 \psi_{\tau\tau}^1 - 2\omega B^1 \psi_{\tau\tau}^2 \\ & - 2\omega D^1 \psi_{K\tau}^2 - 2B^1 D^1 \psi_{K\tau}^1 - 2\omega D^2 \psi_{K\tau}^1 - (D^1)^2 \psi_{KK}^1 \\ & - D^1 D_K^1 \psi_K^1 - 2B^1 \psi_{\tau r}^1 \psi_r^1 - 2\omega \psi_{\tau r}^2 \psi_r^1 - 2D^1 \psi_{Kr}^1 \psi_r^1 \\ & - 2\omega \psi_{\tau r}^1 \psi_r^2 - 2B^1 \psi_{\tau\theta}^1 \psi_\theta^1 / r^2 - 2\omega \psi_{\tau\theta}^2 \psi_\theta^1 / r^2 \\ & - 2D^1 \psi_{K\theta}^1 \psi_\theta^1 / r^2 - 2\omega \psi_{\tau\theta}^1 \psi_\theta^2 / r^2 - 2B^1 \psi_{\tau z}^1 \psi_z^1 \\ & - 2\omega \psi_{\tau z}^2 \psi_z^1 - 2D^1 \psi_{Kz}^1 \psi_z^1 - 2\omega \psi_{\tau z}^1 \psi_z^2 \\ & + 2\omega^2 B^1 \psi_{\tau\tau z}^1 \psi_\tau^1 / g + \omega^3 \psi_{\tau\tau z}^2 \psi_\tau^1 / g + 2\omega^2 D^1 \psi_{K\tau z}^1 \psi_\tau^1 / g \\ & + \omega^3 \psi_{\tau\tau z}^1 \psi_\tau^2 / g + \omega^2 B^1 \psi_{\tau\tau z}^1 \psi_\tau^1 / g + \omega^2 D^1 \psi_{\tau\tau z}^1 \psi_K^1 / g \\ & + \omega \psi_\tau^1 \psi_{zz}^2 + \omega \psi_\tau^2 \psi_{zz}^1 + B^1 \psi_\tau^1 \psi_{zz}^1 + D^1 \psi_K^1 \psi_{zz}^1 \\ & - (\psi_r^1)^2 \psi_{rr}^1 - (\psi_\theta^1)^2 \psi_{\theta\theta}^1 / r^4 - (\psi_z^1)^2 \psi_{zz}^1 + (\psi_\theta^1)^2 \psi_r^1 / r^3 \\ & - 2\psi_r^1 \psi_z^1 \psi_{rz}^1 - 2\psi_r^1 \psi_\theta^1 \psi_{r\theta}^1 / r^2 - \psi_z^1 \psi_\theta^1 \psi_{\theta z}^1 / r^2 \\ & + \omega^2 \psi_{\tau\tau z}^1 (\psi_r^1)^2 / (2g) + \omega^2 (\psi_\theta^1)^2 \psi_{\tau\tau z}^1 / (2gr^2) \end{aligned}$$

$$\begin{aligned}
& + \omega^2 \psi_{\tau\tau Z}^1 (\psi_Z^1)^2 / (2g) + (\psi_r^1)^2 \psi_{ZZ}^1 / 2 + (\psi_\theta^1)^2 \psi_{ZZ}^1 / (2r^2) \\
& + (\psi_Z^1)^2 \psi_{ZZ}^1 / 2 + 2\omega^2 \psi_\tau^1 \psi_{rZ}^1 \psi_{\tau r}^1 / g + 2\omega^2 \psi_\tau^1 \psi_{\theta Z}^1 \psi_{\tau\theta}^1 / g \\
& + 2\omega^2 \psi_\tau^1 \psi_{\theta Z}^1 \psi_{\tau\theta}^1 / (gr^2) + 2\omega^2 \psi_\tau^1 \psi_{\theta Z}^1 \psi_{\tau\theta Z}^1 / (gr^2) \\
& + 2\omega^2 \psi_\tau^1 \psi_{ZZ}^1 \psi_{\tau Z}^1 / g + 2\omega^2 \psi_\tau^1 \psi_Z^1 \psi_{\tau ZZ}^1 / g \\
& - \omega^4 \psi_\tau^1 \psi_{\tau Z}^1 \psi_{\tau\tau Z}^1 / g^2 - \omega^2 \psi_{\tau Z}^1 \psi_{\tau}^1 \psi_{ZZ}^1 / g \\
& - \omega^4 (\psi_\tau^1)^2 \psi_{\tau\tau ZZ}^1 / (2g^2) - \omega^2 (\psi_\tau^1)^2 \psi_{ZZZ}^1 / (2g) \quad .
\end{aligned} \tag{4.23}$$

As expressed by (4.6), $\psi^{(i)}$ satisfies the differential equation and the boundary condition at the container wall F . Thus, if the terms $A_{pm}^{(i)}$ are chosen so that the boundary condition on the free surface equations (4.21), (4.22), and (4.23) are satisfied, then the nonlinear fluid oscillation problem has been solved through the third-order terms.

Since this analysis is to determine a solution near the first non-axisymmetric mode of oscillation, ω is taken to be defined by

$$\omega = \omega_{11} \tag{4.24}$$

First-Order Solution

It is clear from the linear solution shown previously that equation (4.21) is satisfied by choosing

$$\psi^1 = K \cos(\tau) \phi_{11} \quad (4.25)$$

The wave height η^1 can be obtained from equation (2.23d) after equations (4.15) and (4.25) have been used.

Second-Order Solution

To determine an expression for A_{pm}^2 so that the second-order free surface boundary condition is satisfied, equations (4.6) and (4.24) are substituted in equation (4.22) to yield

$$\begin{aligned} & \sum_{p=0}^{\infty} \sum_{m=1}^{\infty} \{ [\omega^2 (A_{pm}^2)_{\tau\tau} + \omega_{pm}^2 A_{pm}^2] \hat{\phi}_{pm} \cos(p\theta) \} \\ &= \sin(2\tau) \{ \omega K^2 [(\hat{\phi}_{11})_r]^2/2 + \omega K^2 (\hat{\phi}_{11})^2/(2r^2) + \omega^5 K^2 (\hat{\phi}_{11})^2/(2g^2) \} \\ &+ \cos(\theta) [2B^1 \omega K \hat{\phi}_{11} \cos(\tau) + 2\omega D^1 K \hat{\phi}_{11} \sin(\tau)] \\ &+ \cos(2\theta) \sin(2\tau) \{ \omega K^2 [(\hat{\phi}_{11})_r]^2/2 + \omega^2 K (\hat{\phi}_{11})^2/(2r^2) \\ &+ \omega^5 K^2 (\hat{\phi}_{11})^2/(2g^2) \} \quad , \end{aligned} \quad (4.26)$$

where the identities

$$\cos^2(p\theta) = [1 + \cos(2p\theta)]/2 \quad (4.27)$$

$$\sin^2(p\theta) = [1 - \cos(2p\theta)]/2 \quad (4.28)$$

$$\sin(2\tau) = 2 \sin(\tau) \cos(\tau) \quad (4.29)$$

and the expressions

$$\phi_{pm} = \hat{\phi}_{pm} \cos(p\theta) \quad (4.30)$$

$$(\phi_{pm})_z = \omega_{pm}^2 \phi_{pm}/g \quad (4.31)$$

have been used.

The two summations in equation (4.26) can be removed by using the orthogonality relationship (3.24) after the equation has been multiplied by ϕ_{qn} and integrated over the surface S_0 . The result is

$$\begin{aligned} \omega^2 (A_{qn}^2)_{\tau\tau} + \omega_{qn}^2 A_{qn}^2 &= 2B^1 \omega K \cos(\tau) \delta_{q1} \delta_{n1} + 2\omega D^1 \sin(\tau) \delta_{q1} \delta_{n1} \\ &+ \pi \omega K^2 \sin(2\tau) (I_{qn}^1 + I_{qn}^2 + \omega^4 I_{qn}^3 / g^2) \delta_{q0} \\ &+ \pi \omega K^2 \sin(2\tau) (I_{qn}^1 - I_{qn}^2 \\ &\quad + \omega^4 I_{qn}^3 / g^2) \delta_{q2} / 2, \end{aligned} \quad (4.32)$$

where

$$I_{qn}^1 = \int_0^R \hat{\phi}_{qn} [(\hat{\phi}_{11})_r]^2 r dr \quad (4.33)$$

$$I_{qn}^2 = \int_0^R \hat{\phi}_{qn} (\hat{\phi}_{11})^2 r^{-1} dr \quad (4.34)$$

$$I_{qn}^3 = \int_0^R \hat{\phi}_{qn} (\hat{\phi}_{11})^2 r dr \quad (4.35)$$

Substituting the Fourier expansion of A_{qn}^2

$$A_{qn}^2 = \frac{1}{2} a_{qn}^0 + \sum_{\ell=1}^{\tau} [a_{qn}^{\ell} \cos(\ell\tau) + b_{qn}^{\ell} \sin(\ell\tau)] \quad (4.36)$$

in equation (4.32), it becomes

$$\begin{aligned} & \omega_{qn}^2 a_{qn}^0 / 2 + \sum_{\ell=1}^{\infty} [(\omega_{qn}^2 - \ell^2 \omega^2) a_{qn}^{\ell} \cos(\ell\tau) + (\omega_{qn}^2 - \ell^2 \omega^2) b_{qn}^{\ell} \sin(\ell\tau)] \\ &= \delta_{q1} \delta_{n1} 2\omega [KB^1 \cos(\tau) + D^1 \sin(\tau)] + \pi \omega K^2 \delta_{q0} (I_{qn}^1 + I_{qn}^2 \\ & \quad + \omega^4 I_{qn}^3 / g^2) \sin(2\tau) \\ & \quad (1/2) \pi \omega K^2 \delta_{q2} (I_{qn}^1 - I_{qn}^2 + \omega^4 I_{qn}^3 / g^2) \sin(2\tau) \end{aligned} \quad (4.37)$$

The Fourier coefficients of the expansion for A_{qn}^2 can be determined from equation (4.37) by equating the coefficients of $\cos(\ell\tau)$ and $\sin(\ell\tau)$ after the conditions expressed by equations (4.9) and (4.10) have been imposed and the assumption made that $\omega_{qn}^2 \neq \ell^2 \omega^2$. This assumption will be examined later. Thus,

$$B^1 = 0 \quad (4.38)$$

$$D^1 = 0 \quad (4.39)$$

$$b_{0n}^2 = \pi \omega K^2 (I_{0n}^1 + I_{0n}^2 + \omega^4 I_{0n}^3 / g^2) / (\omega_{0n}^2 - 4\omega^2) \quad (4.40)$$

$$b_{2n}^2 = (1/2) \pi \omega^2 K^2 (I_{2n}^1 - I_{2n}^2 + \omega^4 I_{2n}^3 / g^2) / (\omega_{2n}^2 - 4\omega^2) \quad (4.41)$$

$$a_{qn}^{\ell} = 0 \quad (4.42)$$

$$b_{qn}^{\ell} = 0 \quad \ell \neq 2 \quad (4.43)$$

$$b_{qn}^2 = 0 \quad q \neq 0 \quad \text{or} \quad q \neq 2 \quad (4.44)$$

When equations (4.38) through (4.44) are used, the second-order term in the expansion of the velocity potential becomes

$$\psi^2 = \sin(2\tau) \sum_{n=1}^{\infty} (b_{0n}^2 \phi_{0n} + b_{2n}^2 \phi_{2n}) \quad (4.45)$$

The wave height η^2 can be evaluated from equation (2.24d) after equations (4.15), (4.16), (4.25), and (4.45) have been used.

Third-Order Solution

The third-order solution is developed in a manner similar to that used for the second-order solution. Equations (4.6), (4.25), and (4.45) are substituted in the third-order free surface boundary condition (4.23) to yield

$$\begin{aligned} & \sum_{p=0}^{\infty} \sum_{m=1}^{\infty} \{ [\omega^2 (A_{pm}^3)_{\tau\tau} + \omega_{pm}^2 A_{pm}^3] \hat{\phi}_{pm} \cos(p\theta) \} \\ &= \cos(\tau) \cos(\theta) \left\{ 2\omega KB^2 \hat{\phi}_{11} - \omega K(\hat{\phi}_{11})_r \sum_{j=1}^{\infty} b_{0j}^2 (\hat{\phi}_{0j})_r \right. \\ & \quad \left. - (1/2) \omega K(\hat{\phi}_{11})_r \sum_{j=1}^{\infty} b_{2j}^2 (\hat{\phi}_{2j})_r \right\} \end{aligned}$$

$$\begin{aligned}
& - \omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \hat{\phi}_{2j} \right) / r^2 \\
& + (1/2) \omega^3 K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^2 \hat{\phi}_{2j} \right) / g^2 \\
& - (1/4) \omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^4 \hat{\phi}_{2j} \right) / g^2 \\
& - (9/16) K^3 [(\hat{\phi}_{11})_r]^2 (\hat{\phi}_{11})_{rr} + (3/16) K^3 \omega^8 (\hat{\phi}_{11})^3 / g^4 \\
& - (3/8) K^3 \omega^4 [(\hat{\phi}_{11})_r]^2 \hat{\phi}_{11} / g^2 + (3/16) K^3 (\hat{\phi}_{11})^3 / r^4 \\
& + (3/16) K^3 (\hat{\phi}_{11})^2 (\hat{\phi}_{11})_r / r^3 - (3/8) K^3 [(\hat{\phi}_{11})_r]^2 \hat{\phi}_{11} / r^2 \\
& - (1/8) K^3 \omega^4 (\hat{\phi}_{11})^3 / (g^2 r^2) \Big\} \\
& + \cos(\tau) \cos(3\theta) \Big\{ - (1/2) \omega K (\hat{\phi}_{11})_r \sum_{j=1}^{\infty} b_{2j}^2 (\hat{\phi}_{2j})_r \\
& + \omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \hat{\phi}_{2j} \right) / r^2 \\
& + (1/2) \omega^3 K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^2 \hat{\phi}_{2j} \right) / g^2 \\
& - (1/4) \omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^4 \hat{\phi}_{2j} \right) / g^2 \\
& - (3/16) K^3 [(\hat{\phi}_{11})_r]^2 (\hat{\phi}_{11})_{rr} \\
& + (1/16) K^3 \omega^8 (\hat{\phi}_{11})^3 / g^4 \\
& - (1/8) K^3 \omega^4 [(\hat{\phi}_{11})_r]^2 \hat{\phi}_{11} / g^2 - (3/16) K^3 (\hat{\phi}_{11})^3 / r^4
\end{aligned}$$

$$\begin{aligned}
& - (3/16) K^3 (\hat{\phi}_{11})^2 (\hat{\phi}_{11})_r / r^3 \\
& + (3/8) K^3 [(\hat{\phi}_{11})_r]^2 \hat{\phi}_{11} / r^2 \\
& + (1/8) K^3 \omega^4 (\hat{\phi}_{11})^3 / (g^2 r^2) \Big\} \\
& + \cos(3\tau) \cos(\theta) \Big\{ - 3\omega K (\hat{\phi}_{11})_r \sum_{j=1}^{\infty} b_{0j}^2 (\hat{\phi}_{0j})_r \\
& - (3/2) (\hat{\phi}_{11})_r \sum_{j=1}^{\infty} b_{2j}^2 (\hat{\phi}_{2j})_r \\
& - 3\omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \hat{\phi}_{2j} \right) / r^2 \\
& - (5/2) \omega^3 K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^2 \hat{\phi}_{2j} \right) / g^2 \\
& - 5\omega^3 K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{0j}^2 \omega_{0j}^2 \hat{\phi}_{0j} \right) / g^2 \\
& + (1/2) \omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{0j}^2 \omega_{0j}^4 \hat{\phi}_{0j} \right) / g^2 \\
& + (1/4) \omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^4 \hat{\phi}_{2j} \right) / g^2 \\
& - (3/16) K^3 [(\hat{\phi}_{11})_r]^2 (\hat{\phi}_{11})_{rr} \\
& - (15/16) K^3 \omega^8 (\hat{\phi}_{11})^3 / g^4 \\
& - (9/8) K^3 \omega^4 \hat{\phi}_{11} [(\hat{\phi}_{11})_r]^2 / g^2 + (1/16) K^3 (\hat{\phi}_{11})^3 / r^4 \\
& + (1/16) K^3 (\hat{\phi}_{11})^2 (\hat{\phi}_{11})_r / r^3
\end{aligned}$$

$$\begin{aligned}
& - (1/8) K^3 [(\hat{\phi}_{11})_{\mathbf{r}}]^2 \hat{\phi}_{11} / r^2 \\
& - (3/8) K^3 \omega^4 (\hat{\phi}_{11})^3 / (r^2 g^2) \Big\} \\
+ \cos(3\tau) \cos(3\theta) \Big\{ & - (3/2) \omega K (\hat{\phi}_{11})_{\mathbf{r}} \sum_{j=1}^{\infty} b_{2j}^2 (\hat{\phi}_{2j})_{\mathbf{r}} \\
& + 3\omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \hat{\phi}_{2j} \right) / r^2 \\
& - (5/2) \omega^3 K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^2 \hat{\phi}_{2j} \right) / g^2 \\
& + (1/4) \omega K \hat{\phi}_{11} \left(\sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^4 \hat{\phi}_{2j} \right) / g^2 \\
& - (1/16) K^3 [(\hat{\phi}_{11})_{\mathbf{r}}]^2 (\hat{\phi}_{11})_{\mathbf{r}\mathbf{r}} \\
& - (5/16) K^3 \omega^8 (\hat{\phi}_{11})^3 / g^4 \\
& - (3/8) \omega^4 K^3 [(\hat{\phi}_{11})_{\mathbf{r}}]^2 \hat{\phi}_{11} / g^2 \\
& - (1/16) K^3 (\hat{\phi}_{11})^3 / r^4 - (1/16) K^3 (\hat{\phi}_{11})^2 (\hat{\phi}_{11})_{\mathbf{r}} / r^3 \\
& + (1/8) K^3 [(\hat{\phi}_{11})_{\mathbf{r}}]^2 \hat{\phi}_{11} / r^2 \\
& (3/8) K^3 \omega^4 (\hat{\phi}_{11})^3 / (g^2 r^2) \Big\} \\
+ \sin(\tau) \cos(\theta) (2\omega D^2 \hat{\phi}_{11}) & \quad . \tag{4.46}
\end{aligned}$$

The two summations in equation (4.46) can be removed by using the orthogonality relationship (3.24) after the equation has been multiplied by $\phi_{\mathbf{qn}}$ and integrated over the surface S_0 . The result is

$$\begin{aligned}
& \omega^2(A_{qn}^3)_{\tau\tau} + \omega_{qn}^2 A_{qn}^3 \\
& = \cos(\tau) \{ 2\omega KB^2 \delta_{q1} \delta_{n1} + \delta_{q1} \pi [-\omega KI_{qn}^4 - (1/2)\omega KI_{qn}^5 - \omega KI_{qn}^6 \\
& \quad + (1/2)\omega^3 KI_{qn}^7/g^2 + \omega^3 KI_{qn}^8/g^2 \\
& \quad - (1/2)\omega KI_{qn}^9/g^2 - (1/4)\omega KI_{qn}^{10}/g^2 \\
& \quad - (9/16)K^3 I_{qn}^{11} + (3/16)K^3 \omega^8 I_{qn}^{13}/g^4 \\
& \quad - (3/8)K^3 \omega^4 I_{qn}^{15}/g^2 + (3/16)K^3 I_{qn}^{12} \\
& \quad + (3/16)K^3 I_{qn}^{14} - (3/8)K^3 I_{qn}^{16} \\
& \quad - (1/8)K^3 \omega^4 I_{qn}^{17}/g^2] \\
& \quad + \delta_{q3} \pi [- (1/2)\omega KI_{qn}^5 + \omega KI_{qn}^6 + (1/2)\omega^3 KI_{qn}^7/g^2 \\
& \quad - (1/4)\omega KI_{qn}^{10}/g^2 - (3/16)K^3 I_{qn}^{11} \\
& \quad + (1/16)K^3 \omega^8 I_{qn}^{13}/g^4 - (1/8)K^3 \omega^4 I_{qn}^{15}/g^2 \\
& \quad - (3/16)K^3 I_{qn}^{12} - (3/16)K^3 I_{qn}^{14} + (3/8)K^3 I_{qn}^{16} \\
& \quad + (1/8)K^3 \omega^4 I_{qn}^{17}/g^2] \} \\
& + \cos(3\tau) \{ \delta_{q1} \pi [-3\omega KI_{qn}^4 - (3/2)\omega KI_{qn}^5 - 3\omega KI_{qn}^6 \\
& \quad - (5/2)\omega^3 KI_{qn}^7/g^2 - 5\omega^3 KI_{qn}^8/g^2 \\
& \quad + (1/2)\omega KI_{qn}^9/g^2 + (1/4)\omega KI_{qn}^{10}/g^2 \\
& \quad - (3/16)K^3 I_{qn}^{11} - (15/16)K^3 \omega^8 I_{qn}^{13}/g^4 \\
& \quad - (9/8)K^3 \omega^4 I_{qn}^{15}/g^2 + (1/16)K^3 I_{qn}^{12}
\end{aligned}$$

$$\begin{aligned}
& + (1/16)K^3 I_{qn}^{14} - (1/8)K^3 I_{qn}^{16} - (3/8)K^3 \omega^4 I_{qn}^{17}/g^2] \\
& + \delta_{q3} \pi [- (3/2) \omega K I_{qn}^5 + 3 \omega K I_{qn}^6 - (5/2) \omega^3 K I_{qn}^7 /g^2 \\
& + (1/4) \omega K I_{qn}^{10}/g^2 - (1/16)K^3 I_{qn}^{11} \\
& - (5/16)K^3 \omega^8 I_{qn}^{13}/g^4 - (3/8) \omega^4 K^3 I_{qn}^{15}/g^2 \\
& - (1/16)K^3 I_{qn}^{12} - (1/16)K^3 I_{qn}^{14} + (1/8)K^3 I_{qn}^{16} \\
& + (3/8)K^3 \omega^4 I_{qn}^{17}/g^2] \} \\
& + \sin(\tau) 2\omega D^2 \delta_{q1} \delta_{n1} \quad , \tag{4.47}
\end{aligned}$$

where

$$I_{qn}^4 = \int_0^R \hat{\phi}_{qn}(\hat{\phi}_{11})_r \sum_{j=1}^{\infty} b_{0j}^2(\hat{\phi}_{0j})_r r dr \tag{4.48}$$

$$I_{qn}^5 = \int_0^R \hat{\phi}_{qn}(\hat{\phi}_{11})_r \sum_{j=1}^{\infty} b_{2j}^2(\hat{\phi}_{2j})_r r dr \tag{4.49}$$

$$I_{qn}^6 = \int_0^R \hat{\phi}_{qn} \hat{\phi}_{11} \sum_{j=1}^{\infty} b_{2j} \hat{\phi}_{2j} r^{-1} dr \tag{4.50}$$

$$I_{qn}^7 = \int_0^R \hat{\phi}_{qn} \hat{\phi}_{11} \sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^2 \hat{\phi}_{2j} r dr \tag{4.51}$$

$$I_{qn}^8 = \int_0^R \hat{\phi}_{qn} \hat{\phi}_{11} \sum_{j=1}^{\infty} b_{0j}^2 \omega_{0j}^2 \hat{\phi}_{0j} r dr \tag{4.52}$$

$$I_{qn}^9 = \int_0^R \hat{\phi}_{qn} \hat{\phi}_{11} \sum_{j=1}^{\infty} b_{0j}^2 \omega_{0j}^4 \hat{\phi}_{0j} r dr \quad (4.53)$$

$$I_{qn}^{10} = \int_0^R \hat{\phi}_{qn} \hat{\phi}_{11} \sum_{j=1}^{\infty} b_{2j}^2 \omega_{2j}^4 \hat{\phi}_{2j} r dr \quad (4.54)$$

$$I_{qn}^{11} = \int_0^R \hat{\phi}_{qn} (\hat{\phi}_{11})_{rr} [(\hat{\phi}_{11})_r]^2 r dr \quad (4.55)$$

$$I_{qn}^{12} = \int_0^R \hat{\phi}_{qn} (\hat{\phi}_{11})^3 r^{-3} dr \quad (4.56)$$

$$I_{qn}^{13} = \int_0^R \hat{\phi}_{qn} (\hat{\phi}_{11})^3 r dr \quad (4.57)$$

$$I_{qn}^{14} = \int_0^R \hat{\phi}_{qn} (\hat{\phi}_{11})^2 (\hat{\phi}_{11})_r r^{-2} dr \quad (4.58)$$

$$I_{qn}^{15} = \int_0^R \hat{\phi}_{qn} \hat{\phi}_{11} [(\hat{\phi}_{11})_r]^2 r dr \quad (4.59)$$

$$I_{qn}^{16} = \int_0^R \hat{\phi}_{qn} \hat{\phi}_{11} [(\hat{\phi}_{11})_r]^2 r^{-1} dr \quad (4.60)$$

$$I_{qn}^{17} = \int_0^R \hat{\phi}_{qn} (\hat{\phi}_{11})^3 r^{-1} dr \quad (4.61)$$

Substituting the Fourier expansion of A_{qn}^3

$$A_{qn}^3 = (1/2)c_{qn}^0 + \sum_{\ell=1}^{\infty} [c_{qn}^{\ell} \cos(\ell \tau) + d_{qn}^{\ell} \sin(\ell \tau)] \quad , \quad (4.62)$$

in equation (4.47), the following expression is obtained:

$$\begin{aligned} & (1/2)c_{qn}^0 \omega_{qn}^2 + \sum_{\ell=1}^{\infty} [(\omega_{qn}^2 - \ell^2 \omega^2) c_{qn}^{\ell} \cos(\ell \tau) \\ & + (\omega_{qn}^2 - \ell^2 \omega^2) d_{qn}^{\ell} \sin(\ell \tau)] \\ & = \cos(\tau) \{ 2\omega K B^2 \delta_{q1} \delta_{n1} + \delta_{q1} \pi [-\omega K I_{qn}^4 - (1/2)\omega K I_{qn}^5 - \omega K I_{qn}^6 \\ & \quad + (1/2)\omega^3 K I_{qn}^7 / g^2 + \omega^3 K I_{qn}^8 / g^2 \\ & \quad - (1/2)\omega K I_{qn}^9 / g^2 - (1/4)\omega K I_{qn}^{10} / g^2 \\ & \quad - (9/16) K^3 I_{qn}^{11} + (3/16) K^3 \omega^8 I_{qn}^{13} / g^4 \\ & \quad - (3/8) K^3 \omega^4 I_{qn}^{15} / g^2 + (3/16) K^3 I_{qn}^{12} \\ & \quad + (3/16) K^3 I_{qn}^{14} - (3/8) K^3 I_{qn}^{16} \\ & \quad - (1/8) K^3 \omega^4 I_{qn}^{17} / g^2] \\ & \quad + \delta_{q3} \pi [- (1/2)\omega K I_{qn}^5 + \omega K I_{qn}^6 + (1/2)\omega^3 K I_{qn}^7 / g^2 \\ & \quad - (1/4)\omega K I_{qn}^{10} / g^2 - (3/16) K^3 I_{qn}^{11} \\ & \quad + (1/16) K^3 \omega^8 I_{qn}^{13} / g^4 - (1/8) K^3 \omega^4 I_{qn}^{15} / g^2 \end{aligned}$$

$$\begin{aligned}
& - (3/16) K^3 I_{qn}^{12} - (3/16) K^3 I_{qn}^{14} + (3/8) K^3 I_{qn}^{16} \\
& + (1/8) K^3 \omega^4 I_{qn}^{17} / g^2] \} \\
& + \cos(3\tau) \{ \delta_{q1} \pi [- 3\omega K I_{qn}^4 - (3/2) \omega K I_{qn}^5 - 3\omega K I_{qn}^6 \\
& - (5/2) \omega^3 K I_{qn}^7 / g^2 - 5\omega^3 K I_{qn}^8 / g^2 \\
& + (1/2) \omega K I_{qn}^9 / g^2 + (1/4) \omega K I_{qn}^{10} / g^2 \\
& - (3/16) K^3 I_{qn}^{11} - (15/16) K^3 \omega^8 I_{qn}^{13} / g^4 \\
& - (9/8) K^3 \omega^4 I_{qn}^{15} / g^2 + (1/16) K^3 I_{qn}^{12} \\
& + (1/16) K^3 I_{qn}^{14} - (1/8) K^3 I_{qn}^{16} - (3/8) K^3 \omega^4 I_{qn}^{17} / g^2] \\
& + \delta_{q3} \pi [- (3/2) \omega K I_{qn}^5 + 3\omega K I_{qn}^6 - (5/2) \omega^3 K I_{qn}^7 / g^2 \\
& + (1/4) \omega K I_{qn}^{10} / g^2 - (1/16) K^3 I_{qn}^{11} \\
& - (5/16) K^3 \omega^8 I_{qn}^{13} / g^4 - (3/8) \omega^4 K^3 I_{qn}^{15} / g^2 \\
& - (1/16) K^3 I_{qn}^{12} - (1/16) K^3 I_{qn}^{14} + (1/8) K^3 I_{qn}^{16} \\
& + (3/8) K^3 \omega^4 I_{qn}^{17} / g^2] \} \\
& + \sin(\tau) 2\omega D^2 \delta_{q1} \delta_{n1} \tag{4.63}
\end{aligned}$$

The Fourier coefficients of the expansion for A_{qn}^3 can be determined from equation (4.63) by equating the coefficients of $\cos(\ell\tau)$ and $\sin(\ell\tau)$ after the conditions expressed by equations (4.9) and (4.10) have been imposed and the assumption made that $\omega_{qn}^2 \neq \ell^2 \omega^2$. This assumption will be examined later. Thus,

$$\begin{aligned}
c_{1n}^1 = & \pi [-\omega K I_{1n}^4 - (1/2)\omega K I_{1n}^5 - \omega K I_{1n}^6 + (1/2)\omega^3 K I_{1n}^7 / g^2 \\
& + \omega^3 K I_{1n}^8 / g^2 - (1/2)\omega K I_{1n}^9 / g^2 - (1/4)\omega K I_{1n}^{10} / g^2 \\
& - (9/16)K^3 I_{1n}^{11} + (3/16)K^3 I_{1n}^{12} + (3/16)K^3 \omega^8 I_{1n}^{13} / g^4 \\
& + (3/16)K^3 I_{1n}^{14} - (3/8)K^3 \omega^4 I_{1n}^{15} / g^2 - (3/8)K^3 I_{1n}^{16} \\
& - (1/8)K^3 \omega^4 I_{1n}^{17} / g^2] / (\omega_{1n}^2 - \omega^2) \\
& n \neq 1 \tag{4.64}
\end{aligned}$$

$$\begin{aligned}
c_{3n}^1 = & \pi [- (1/2)\omega K I_{3n}^5 + \omega K I_{3n}^6 + (1/2)\omega^3 K I_{3n}^7 / g^2 \\
& - (1/4)\omega K I_{3n}^{10} / g^2 - (3/16)K^3 I_{3n}^{11} - (3/16)K^3 I_{3n}^{12} \\
& + (1/16)K^3 \omega^8 I_{3n}^{13} / g^4 - (3/16)K^3 I_{3n}^{14} - (1/8)K^3 \omega^4 I_{3n}^{15} / g^2 \\
& + (3/8)K^3 I_{3n}^{16} + (1/8)K^3 \omega^4 I_{3n}^{17} / g^2] / (\omega_{3n}^2 - \omega^2) \tag{4.65}
\end{aligned}$$

$$\begin{aligned}
c_{1n}^3 = & \pi [-3\omega K I_{1n}^4 - (3/2)\omega K I_{1n}^5 - 3\omega K I_{1n}^6 - (5/2)\omega^3 K I_{1n}^7 / g^2 \\
& - 5\omega^3 K I_{1n}^8 / g^2 + (1/2)\omega K I_{1n}^9 / g^2 + (1/4)\omega K I_{1n}^{10} \\
& - (3/16)K^3 I_{1n}^{11} + (1/16)K^3 I_{1n}^{12} - (15/16)K^3 \omega^8 I_{1n}^{13} / g^4 \\
& + (1/16)K^3 I_{1n}^{14} - (9/8)K^3 \omega^4 I_{1n}^{15} / g^2 - (1/8)K^3 I_{1n}^{16} \\
& - (3/8)K^3 \omega^4 I_{1n}^{17} / g^2] / (\omega_{1n}^2 - 9\omega^2) \tag{4.66}
\end{aligned}$$

$$\begin{aligned}
c_{3n}^3 = & \pi [- (3/2) \omega K I_{3n}^5 + 3 \omega K I_{3n}^6 - (5/2) \omega^3 K I_{3n}^7 / g^2 \\
& + (1/4) \omega K I_{3n}^{10} / g^2 - (1/16) K^3 I_{3n}^{11} - (1/16) K^3 I_{3n}^{12} \\
& - (5/16) K^3 \omega^8 I_{3n}^{13} / g^4 - (1/16) K^3 I_{3n}^{14} - (3/8) \omega^4 K^3 I_{3n}^{15} / g^2 \\
& + (1/8) K^3 I_{3n}^{16} + (3/8) K^3 \omega^4 I_{3n}^{17} / g^2] / (\omega_{3n}^2 - 9\omega^2)
\end{aligned} \tag{4.67}$$

All values of c_{qn}^ℓ equal zero except the ones given by equations (4.64) through (4.67).

$$d_{qn}^\ell = 0 \tag{4.68}$$

$$\begin{aligned}
B^2 = & \pi [\omega K I_{11}^4 + (1/2) \omega K I_{11}^5 + \omega K I_{11}^6 - (1/2) \omega^3 K I_{11}^7 / g^2 \\
& - \omega^3 K I_{11}^8 / g^2 + (1/2) \omega K I_{11}^9 / g^2 + (1/4) \omega K I_{11}^{10} / g^2 \\
& + (9/16) K^3 I_{11}^{11} - (3/16) K^3 I_{11}^{12} - (3/16) K^3 \omega^8 I_{11}^{13} / g^4 \\
& - (3/16) K^3 I_{11}^{14} + (3/8) K^3 \omega^4 I_{11}^{15} / g^2 + (3/8) K^3 I_{11}^{16} \\
& + (1/8) K^3 \omega^4 I_{11}^{17} / g^2] / (2\omega K)
\end{aligned} \tag{4.69}$$

$$D^2 = 0 \tag{4.70}$$

The third-order term in the expansion of the velocity potential can now be expressed as

$$\begin{aligned}
\psi^3 = & \sum_{n=1}^{\infty} [(c_{1n}^1 \phi_{1n} + c_{3n}^1 \phi_{3n}) \cos(\tau) \\
& + (c_{1n}^3 \phi_{1n} + c_{3n}^3 \phi_{3n}) \cos(3\tau)]
\end{aligned} \tag{4.71}$$

The wave height η^3 can be evaluated from equation (2.25d) once equations (4.15), (4.16), (4.17), (4.25), (4.45), and (4.71) have been used.

A nonlinear solution has been developed through the third-order term, but two unknown quantities, K and ϵ , still exist. To evaluate K and obtain a meaningful physical significance for ϵ , consider the first approximation to the wave height η .

$$\eta = \epsilon \eta^1 = \epsilon K \omega \sin(\tau) \phi_{11}/g \quad \text{on } S_0 \quad (4.72)$$

Since K is arbitrary, let it be evaluated so that

$$\eta^1 = 1 \quad \text{at} \quad \tau = \pi/2, \quad \theta = 0, \quad \text{and} \quad r = R \quad ; \quad (4.73)$$

therefore K becomes

$$K = g/[\omega \phi_{11}(R, 0, 0)] \quad (4.74)$$

and

$$\eta = \epsilon \quad \text{at} \quad \tau = \pi/2, \quad \theta = 0, \quad \text{and} \quad r = R \quad (4.75)$$

Thus the wave height of the linear theory evaluated at $\tau = \pi/2$, $\theta = 0$, and $r = R$ is equal to ϵ . This is the same physical significance found by Mack [4], DiMaggio and Rehm [5], and Baird [6].

It is useful to manipulate the nonlinear frequency $\dot{\tau}$ so that a frequency correction factor can be extracted from it. This is accomplished by squaring $\dot{\tau}$ and rearranging the terms

$$(\dot{\tau})^2 = \omega^2 [1 + (2B^2/\omega) \epsilon^2 + \dots] \quad (4.76)$$

Thus, for a solution through third-order terms, the nonlinear frequency parameter $(\dot{\tau})^2$ is different from the linear frequency parameter by a factor of $2B^2/\omega$, which is called the frequency correction factor G_c . It is also convenient to let

$$(\dot{\tau})^2 = \lambda^* \quad (4.77)$$

and

$$\omega^2 = \lambda \quad (4.78)$$

Then equation (4.76), for a third-order solution, becomes

$$\lambda^* = \lambda (1 + G_c \epsilon^2) \quad , \quad (4.79)$$

where

$$G_c = 2B^2/\omega \quad (4.80)$$

The solution of the nonlinear fluid oscillation problem in an axisymmetric container has been formulated, but some things of interest should be noted. In the expansion of \dot{K} , all the terms were zero. One would expect this since K is an amplitude parameter and should remain constant for a conservative system such as this one. In the expansion of $\dot{\tau}$, it was found that B^1 was zero. Thus, the second-order equations do not affect the frequency of oscillation.

NUMERICAL RESULTS

Numerical results were obtained for finite-amplitude standing waves near the first antisymmetric linear mode in three axisymmetric containers. Since cylindrical tanks with bulkheads of various shapes are in general use, it was considered appropriate to apply the theory developed in this work to obtain numerical results for them. The containers for which results were obtained were a cylindrical tank with an ellipsoidal bulkhead, a cylindrical tank with a conical bulkhead, and a cylindrical tank with a truncated conical bulkhead.

In the case of the cylindrical tank with an ellipsoidal bulkhead, the eccentricity of the bulkhead was considered a parameter and varied accordingly. The eccentricity, e , is taken to mean the ratio of the depth to the radius of the bulkhead, and numerical results were obtained for values of $e = 0.5$, $e = 1.0$, and $e = 2.0$.

The parameter that was varied in the study of the cylindrical tank with a conical bulkhead was the angle between a horizontal line and the generator of the cone. This angle β took on values of 30 deg, 45 deg, and 60 deg.

The third tank geometry studied is similar to the conical bulkhead tank except that the bulkhead is truncated at one-half the depth of the cone. Thus the cylindrical tank, with a truncated conical bulkhead, has a radius at the bottom equal to one-half the radius of the cylindrical portion of the tank. Again, the angle β was parameterized for values of 30 deg, 45 deg, and 60 deg.

It is convenient to nondimensionalize the numerical results in such a way that one is able to compare the effect of the various tank geometries on the results. The two dimensional quantities in this analysis are length and time. Thus, the radius of the cylindrical portion of the tanks, R_t , and the local gravitational acceleration, g , will be used for this purpose. The nondimensional variables of interest are

$$\begin{aligned}\bar{H} &= H/R_t, & \bar{e} &= e, & \bar{\lambda} &= \lambda R_t/g, & \lambda^* &= \lambda^* R_t/g, \\ \bar{\beta} &= \beta, & \bar{G}_c &= G_c R_t^2, & \epsilon &= \epsilon/R_t, & \eta &= \eta/R_t\end{aligned}$$

Linear Results

To obtain the nonlinear solution it is necessary to solve the linear problem first. Thus, the accuracy of the method by which the linear results were achieved must be considered.

It was explained previously that the axisymmetric container being analyzed was generated by rotating a curve γ about the axis of symmetry, and this curve was expressed in parametric form. It was assumed that in

writing a computer program for this problem, the curve γ was made up of several sections, and each of the individual curves was of the form

$$r = a_i \cos(b_i \alpha + c_i) + d_i \alpha^2 + e_i \alpha + f_i \quad (5.1a)$$

$$z = g_i \sin(h_i \alpha + p_i) + q_i \alpha^2 + s_i \alpha + t_i \quad (5.1b)$$

where α is the parameter.

Thus, by using several sets of equations and selecting the proper coefficients in equation (5.1), an axisymmetric container of quite a general shape can be obtained.

The linear solution is obtained by solving the eigenvalue problem, which is represented by equation (3.21) and from which a great deal can be learned about the convergence of the solution. Symbolically, equation (3.21) can be written as

$$\{[A] + [B] - 2\omega^2/g [C]\} \{a\} = 0 \quad (5.2)$$

The assumption was made previously that the velocity potential of an arbitrary axisymmetric container could be approximated by a series of velocity potentials from a cylinder of radius R_c and height H_c , where R_c equals the maximum radius of the arbitrary container, and H_c equals its height. Thus for the containers under study, R_c is the radius at the free surface. Bearing this in mind, matrices $[B]$ and $[C]$ are diagonal. Matrix $[A]$ is obtained by integrating a function of the assumed velocity

potentials, which is zero at R_c , over the container walls. Therefore, if the container being studied was a cylinder, this function would be zero at the container walls, and $[A]$ would become zero. Thus matrix $[A]$ can be called a deviation matrix since it is a measure of the degree to which the container deviates from a cylinder. A careful study of the function that is integrated to obtain $[A]$ can explain something about the convergence of the problem. The function is zero at R_c and becomes larger as r is decreased; also, the function decreases with fluid depth. Thus one would expect excellent convergence for a container whose radius deviated only slightly from R_c near the free surface, even if it was greatly different at larger depths.

A study was made varying the number of terms, NT , in the approximating series as a check of the convergence. The containers used in this study were a cylindrical tank with an ellipsoidal, a conical, and a truncated conical bulkhead. Each was filled to a depth of $H_c = 1.0$. Tables I, II, and III show the resulting eigenvalues to six decimal places. As expected, the best convergence was observed in the cylindrical tank with the truncated conical bulkhead since it was filled above the conical section, and the deviation matrix should have been small. The worst convergence was observed for the conical bulkhead tank since the fluid was in the conical section.

Figure 2 shows a comparison of the first three nonaxisymmetric frequencies of a fluid in a cylindrical tank with a spherical bulkhead with computed and test results obtained by Budiansky [13]. The comparison is excellent, indicating that the method by which the linear solution was obtained is valid.

Table I. Linear Frequencies of a Fluid Contained in a Cylindrical Tank
with an Ellipsoidal Bulkhead for Nine Values of NT

| k = 1 $\bar{H} = 1$ $\bar{e} = 1$ | | | | | | | | | | |
|-----------------------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|-----------------------|
| NT | $\bar{\lambda}_{k1}$ | $\bar{\lambda}_{k2}$ | $\bar{\lambda}_{k3}$ | $\bar{\lambda}_{k4}$ | $\bar{\lambda}_{k5}$ | $\bar{\lambda}_{k6}$ | $\bar{\lambda}_{k7}$ | $\bar{\lambda}_{k8}$ | $\bar{\lambda}_{k9}$ | $\bar{\lambda}_{k10}$ |
| 2 | 1.57883 | 5.28157 | ----- | ----- | ----- | ----- | ----- | ----- | ----- | ----- |
| 3 | 1.57868 | 5.28119 | 8.50676 | ----- | ----- | ----- | ----- | ----- | ----- | ----- |
| 4 | 1.57865 | 5.28108 | 8.50658 | 11.6847 | ----- | ----- | ----- | ----- | ----- | ----- |
| 5 | 1.57863 | 5.28104 | 8.50652 | 11.6846 | 14.8469 | ----- | ----- | ----- | ----- | ----- |
| 6 | 1.57863 | 5.28102 | 8.50649 | 11.6846 | 14.8469 | 18.0018 | ----- | ----- | ----- | ----- |
| 7 | 1.57862 | 5.28101 | 8.50647 | 11.6846 | 14.8468 | 18.0018 | 21.1528 | ----- | ----- | ----- |
| 8 | 1.57862 | 5.28101 | 8.50646 | 11.6845 | 14.8468 | 18.0018 | 21.1527 | 24.3013 | ----- | ----- |
| 9 | 1.57862 | 5.28101 | 8.50646 | 11.6845 | 14.8468 | 18.0018 | 21.1527 | 24.3012 | 27.4482 | ----- |
| 10 | 1.57862 | 5.28100 | 8.50646 | 11.6845 | 14.8468 | 18.0017 | 21.1527 | 24.3012 | 27.4481 | 30.5940 |

Table II. Linear Frequencies of a Fluid Contained in a Cylindrical Tank
with a Conical Bulkhead for Nine Values of NT

| $k = 1 \quad H = 1 \quad \beta = 45^\circ$ | | | | | | | | | | |
|--------------------------------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|-----------------------|
| NT | $\bar{\lambda}_{k1}$ | $\bar{\lambda}_{k2}$ | $\bar{\lambda}_{k3}$ | $\bar{\lambda}_{k4}$ | $\bar{\lambda}_{k5}$ | $\bar{\lambda}_{k6}$ | $\bar{\lambda}_{k7}$ | $\bar{\lambda}_{k8}$ | $\bar{\lambda}_{k9}$ | $\bar{\lambda}_{k10}$ |
| 2 | 1.07943 | 4.85539 | ----- | ----- | ----- | ----- | ----- | ----- | ----- | ----- |
| 3 | 1.06931 | 4.79928 | 8.10620 | ----- | ----- | ----- | ----- | ----- | ----- | ----- |
| 4 | 1.06444 | 4.77214 | 8.10531 | 11.2961 | ----- | ----- | ----- | ----- | ----- | ----- |
| 5 | 1.06168 | 4.75637 | 8.02401 | 11.2460 | 14.4670 | ----- | ----- | ----- | ----- | ----- |
| 6 | 1.05989 | 4.74610 | 8.00515 | 11.2169 | 14.4191 | 17.6286 | ----- | ----- | ----- | ----- |
| 7 | 1.05866 | 4.73897 | 7.99198 | 11.1971 | 14.3906 | 17.5826 | 20.7850 | ----- | ----- | ----- |
| 8 | 1.0577 | 4.73377 | 7.98230 | 11.1825 | 14.3706 | 17.5549 | 20.7406 | 23.9383 | ----- | ----- |
| 9 | 1.05710 | 4.72984 | 7.97494 | 11.1715 | 14.3555 | 17.5351 | 20.7137 | 23.8952 | 27.0894 | ----- |
| 10 | 1.05657 | 4.72676 | 7.96916 | 11.1628 | 14.3438 | 17.5199 | 20.6942 | 23.8690 | 27.0474 | 30.2389 |

Table III. Linear Frequencies of a Fluid Contained in a Cylindrical Tank
with a Truncated Conical Bulkhead for Nine Values of NT

| $k = 1 \quad \bar{H} = 1 \quad \beta = 45^\circ$ | | | | | | | | | | |
|--------------------------------------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|-----------------------|
| NT | $\bar{\lambda}_{k1}$ | $\bar{\lambda}_{k2}$ | $\bar{\lambda}_{k3}$ | $\bar{\lambda}_{k4}$ | $\bar{\lambda}_{k5}$ | $\bar{\lambda}_{k6}$ | $\bar{\lambda}_{k7}$ | $\bar{\lambda}_{k8}$ | $\bar{\lambda}_{k9}$ | $\bar{\lambda}_{k10}$ |
| 2 | 1.68639 | 5.32880 | ----- | ----- | ----- | ----- | ----- | ----- | ----- | ----- |
| 3 | 1.68639 | 5.32880 | 8.53622 | ----- | ----- | ----- | ----- | ----- | ----- | ----- |
| 4 | 1.68639 | 5.32880 | 8.53622 | 11.7060 | ----- | ----- | ----- | ----- | ----- | ----- |
| 5 | 1.68639 | 5.32880 | 8.53622 | 11.7060 | 14.8636 | ----- | ----- | ----- | ----- | ----- |
| 6 | 1.68639 | 5.32880 | 8.53622 | 11.7060 | 14.8636 | 18.0155 | ----- | ----- | ----- | ----- |
| 7 | 1.68639 | 5.32880 | 8.53622 | 11.7060 | 14.8636 | 18.0155 | 21.1644 | ----- | ----- | ----- |
| 8 | 1.68639 | 5.32880 | 8.53622 | 11.7060 | 14.8636 | 18.0155 | 21.1644 | 24.3113 | ----- | ----- |
| 9 | 1.68639 | 5.32880 | 8.53622 | 11.7060 | 14.8636 | 18.0155 | 21.1644 | 24.3113 | 27.4570 | ----- |
| 10 | 1.68639 | 5.32880 | 8.53622 | 11.7060 | 14.8636 | 18.0155 | 21.1644 | 24.3113 | 27.4570 | 30.6019 |

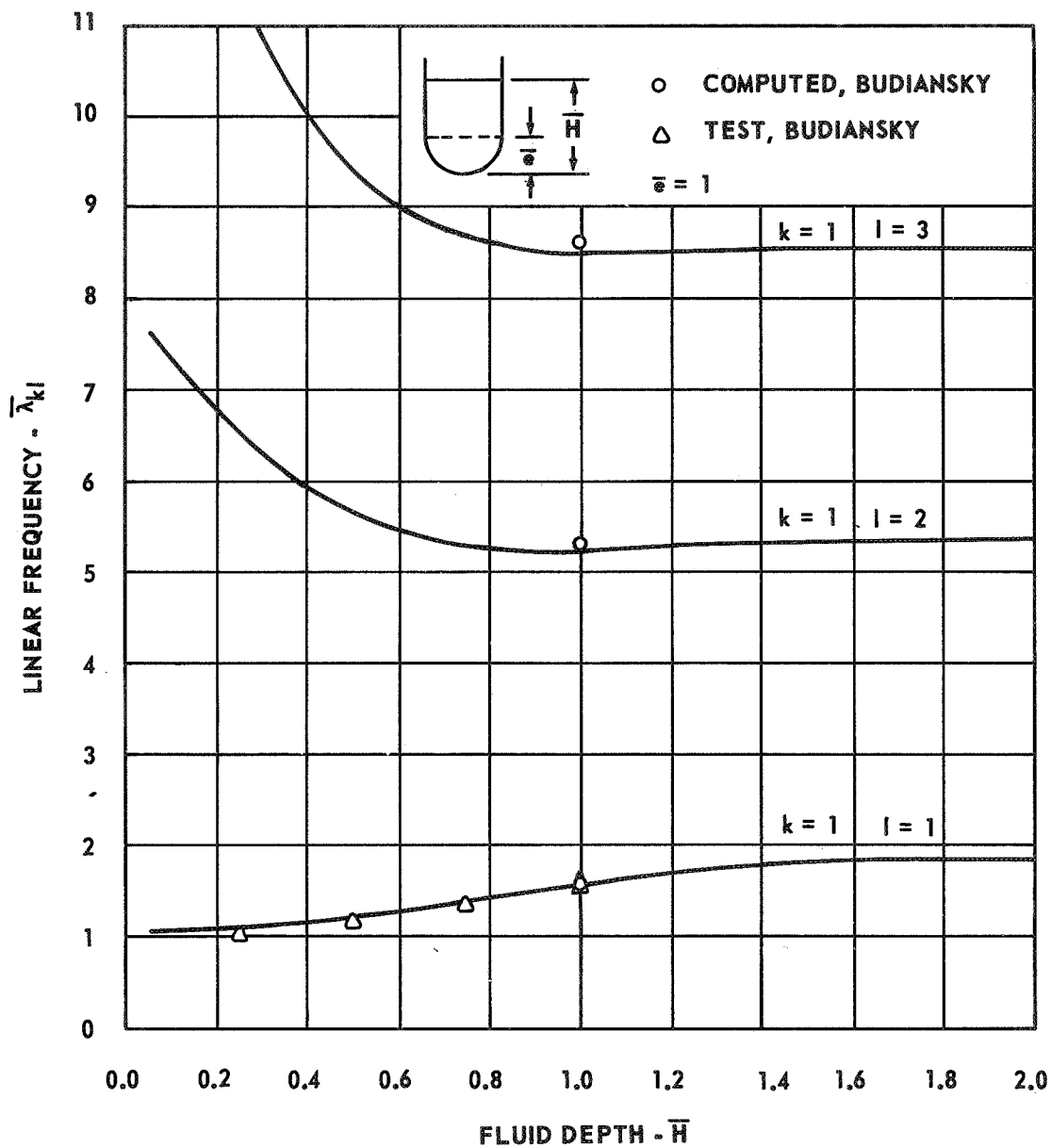


Figure 2. Comparison of Linear Frequencies in a Cylindrical Tank with a Spherical Bulkhead as a Function of Fluid Depth

Figure 3 shows the linear frequency of a fluid in a cylindrical tank with an ellipsoidal bulkhead as a function of depth. Curves are presented for $\bar{e} = 0.5$, $\bar{e} = 1.0$, and $\bar{e} = 2.0$.

Figure 4 shows the linear frequency of a fluid in a cylindrical tank with a conical bulkhead as a function of depth. Curves are presented for $\beta = 30 \text{ deg}$, $\beta = 45 \text{ deg}$, and $\beta = 60 \text{ deg}$. As can be observed, the frequency decreases with depth until the fluid level reaches the conical section, and then it starts to increase.

Figure 5 shows the linear frequency of a fluid in a cylindrical tank with a truncated conical bulkhead as a function of depth. Curves are presented for $\beta = 30 \text{ deg}$, $\beta = 45 \text{ deg}$, and $\beta = 60 \text{ deg}$. In comparison with Figure 4, one notes that they are similar until shallow depths are reached. This difference, of course, is because the flat bottom of the container becomes effective.

It should be noted that as the fluid depth increases, the tank bottom becomes less effective. Thus, as the depth increases, the frequency of the fluid in all three tanks approaches the frequency obtained for a fluid in a cylindrical tank of infinite depth.

Nonlinear Results

The assumption, $\omega_{ij}^2 \neq l^2 \omega^2$, was made previously. At certain critical depths this assumption is violated, and the solution becomes invalid.

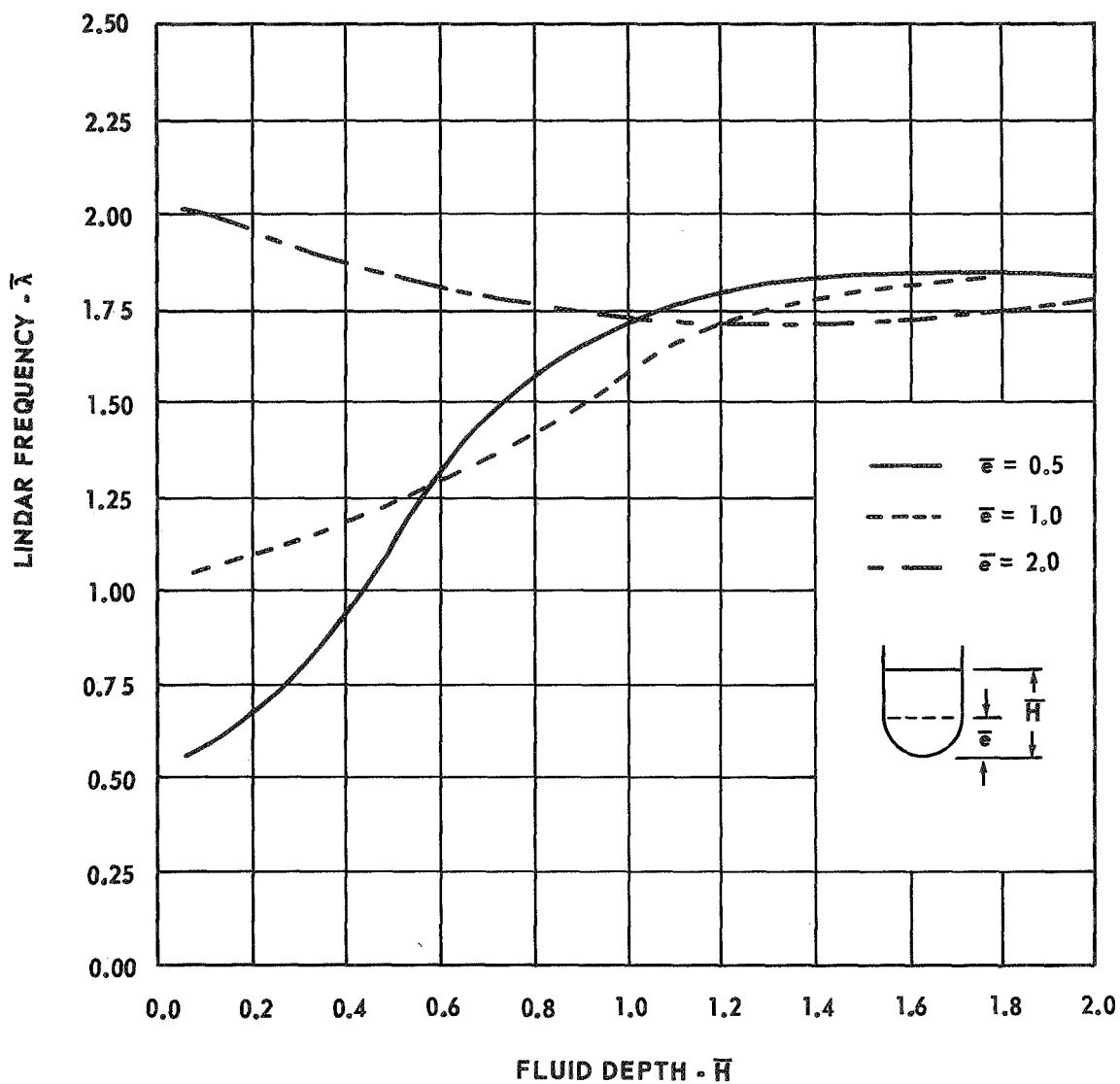


Figure 3. Linear Frequency of a Fluid in a Cylindrical Tank with an Ellipsoidal Bulkhead as Function of Depth

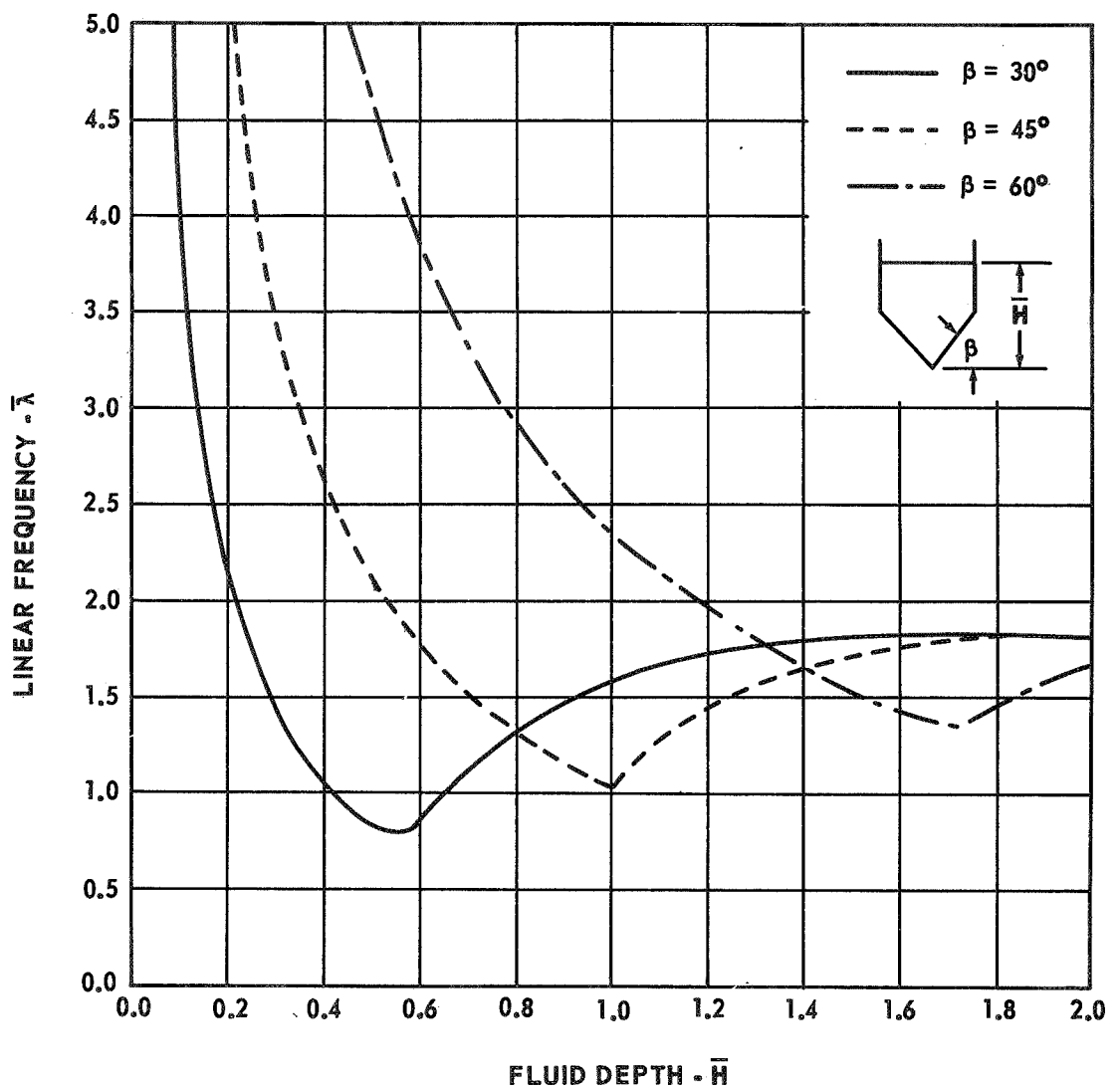


Figure 4. Linear Frequency of a Fluid in a Cylindrical Tank with a Conical Bulkhead as a Function of Depth

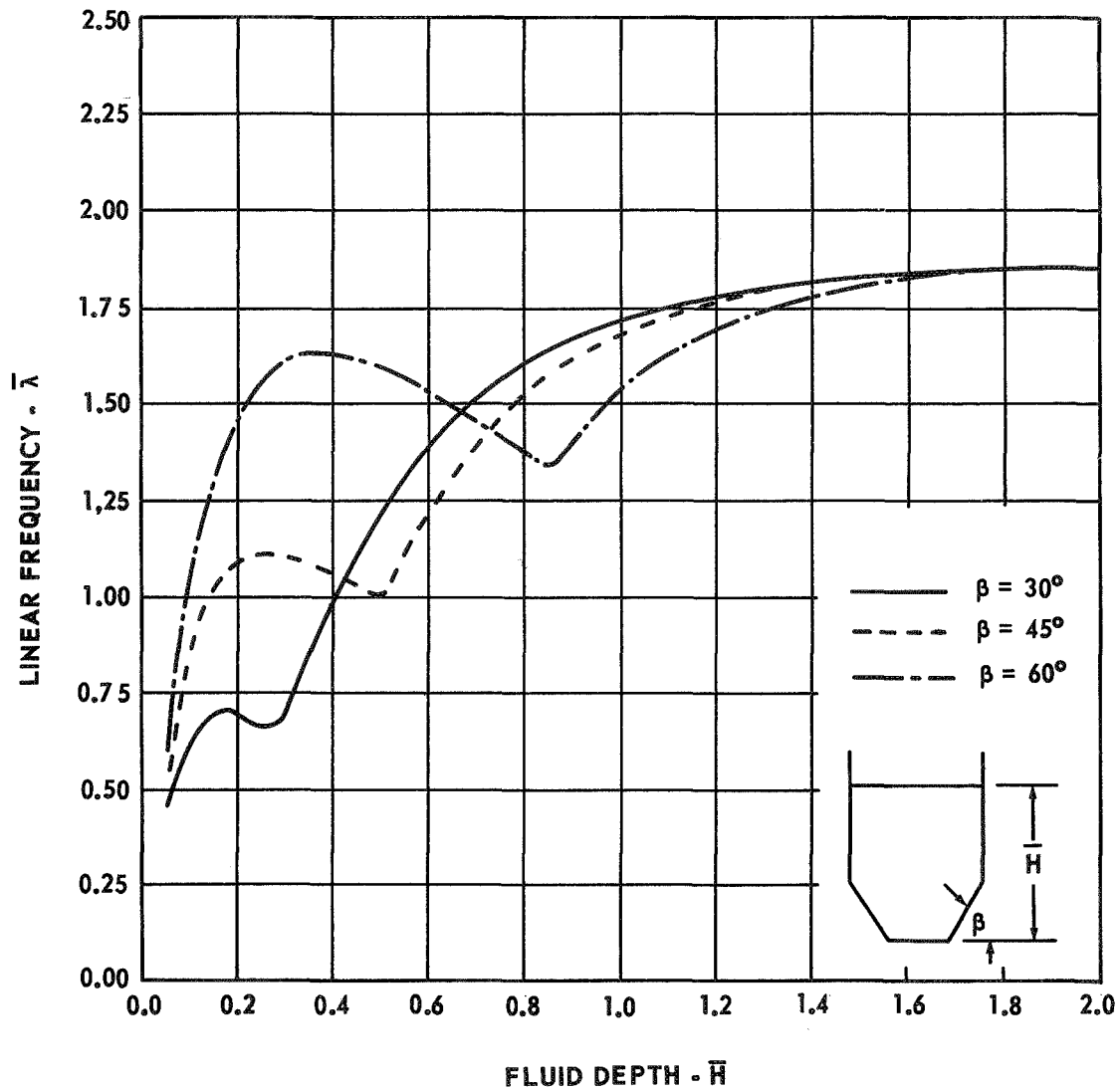


Figure 5. Linear Frequency of a Fluid in a Cylindrical Tank with a Truncated Conical Bulkhead as a Function of Depth

Mack [4] has pointed out the reason for the solution becoming invalid at the critical depths. It has been assumed that there is a first mode of order ϵ oscillating at a frequency ω and that all other modes and harmonics are of order ϵ^2 or higher. However, at the critical depths this assumption is invalid. In general, critical depths do not cause any problem since the solution is known to be invalid and is disregarded. However, at shallow depths it is impossible to do this. Thus, for all cases studied, the solutions were found to be invalid at shallow depths. Therefore, an experimental study should be performed to determine the range for which the solution is valid.

In an attempt to verify this analysis, the frequency correction factor \bar{G}_c was computed for a cylindrical tank as a function of depth and compared with the solution obtained by DiMaggio and Rehm [5] as shown in Figure 6. As can be observed, the comparison is excellent.

Figure 7 shows a comparison of the nonlinear and linear frequency of a fluid in a cylindrical tank with a spherical bulkhead as a function of depth. It can be observed that the nonlinear frequency is lower than the linear frequency above a depth of $H = 0.35$ and higher below this depth.

The linear and nonlinear wave profile was computed for $\bar{\epsilon} = 0.2$ and presented in Figure 8. The curve is shown in a configuration of maximum potential energy, which occurs at $\tau = [(1/2) + n]\pi$ and which is, of course,

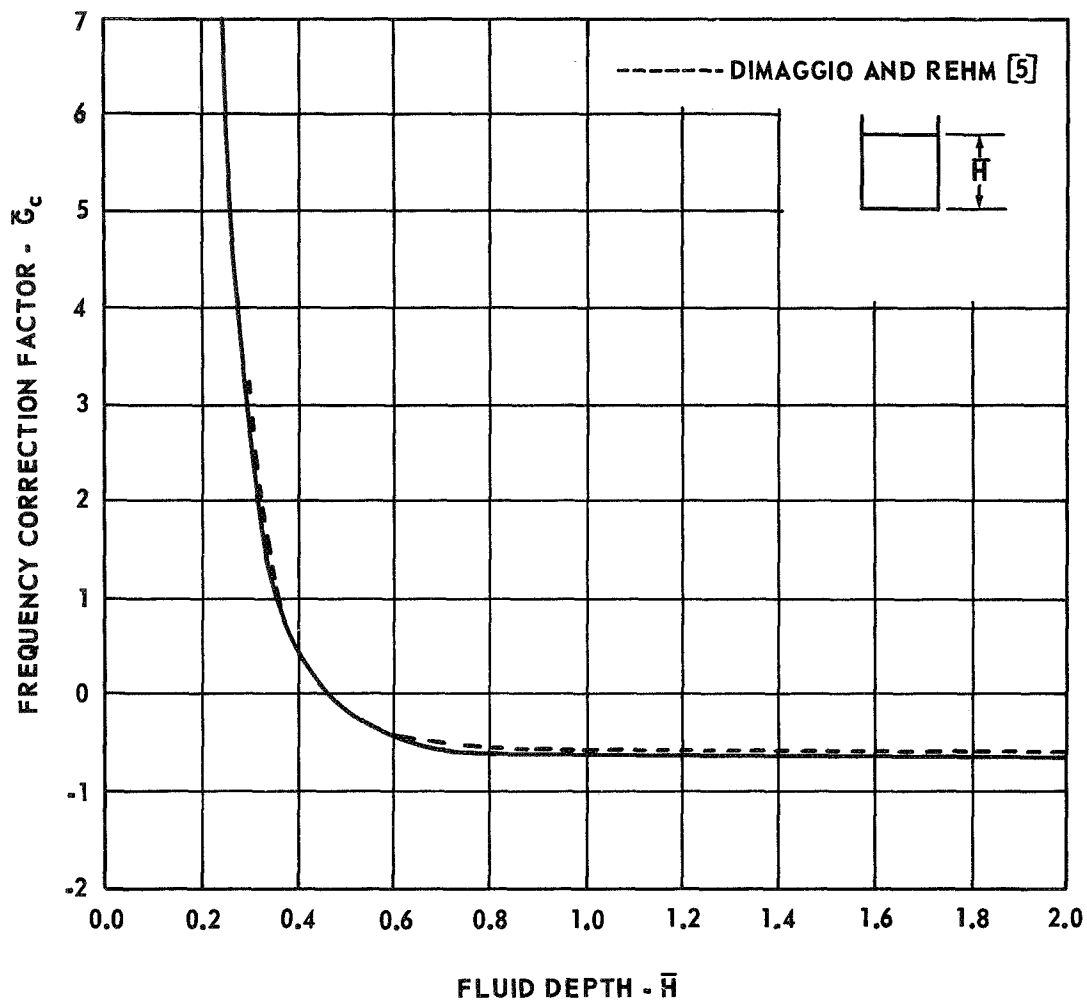


Figure 6. Comparison of Frequency Correction Factor for a Cylindrical Tank as a Function of Depth

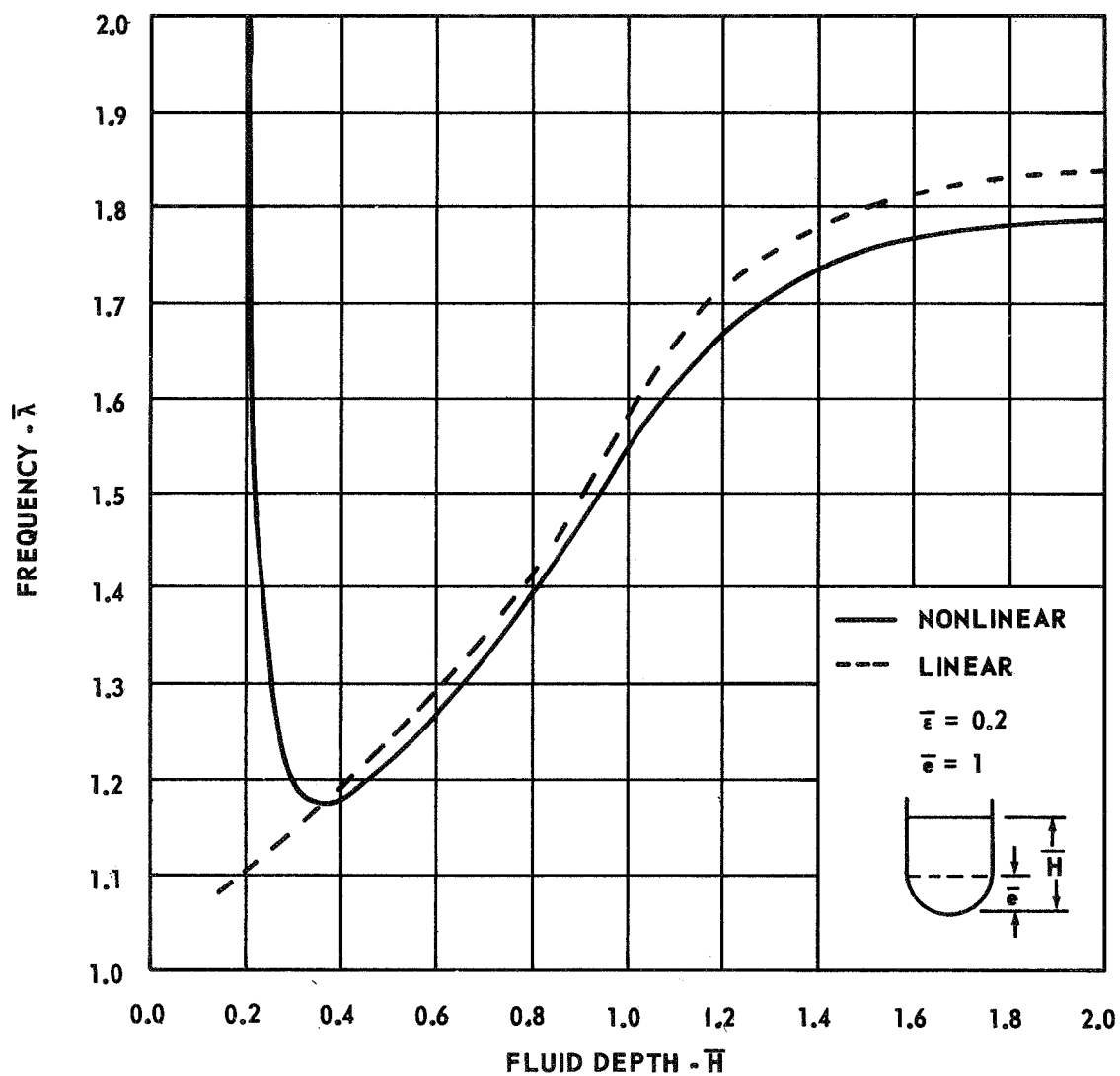


Figure 7. Comparison of Nonlinear and Linear Frequency of a Fluid in a Cylindrical Tank with a Spherical Bulkhead as a Function of Depth

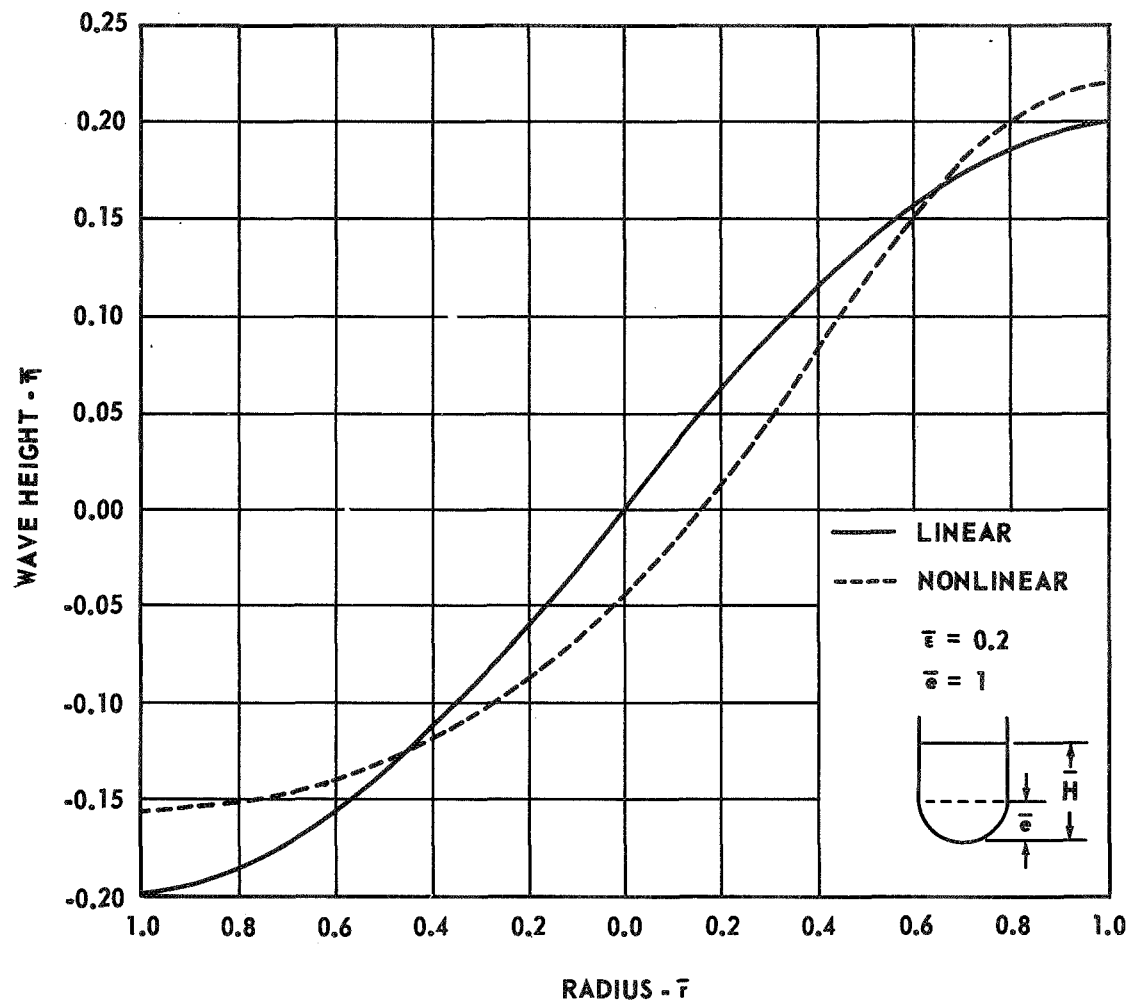


Figure 8. Comparison of Linear and Nonlinear Wave Heights in a Cylindrical Tank with a Spherical Bulkhead as a Function of Radius

when the velocity is zero. The nonlinear wave profile is different from the linear wave in that a nodal line does not exist and the surface is never flat.

Figure 9 shows the frequency correction factor for a fluid in a cylindrical tank with an ellipsoidal bulkhead as a function of depth. It is interesting to note that for $\bar{e} = 0.5$ and $\bar{e} = 1.0$ the curve for \bar{G}_c is similar in shape to the one for a cylinder, but for $\bar{e} = 2.0$ it is quite different. Observing Figure 3, one notes that for $\bar{e} = 2.0$ the linear frequency is increasing with a decrease in depth until the fluid level becomes small. It appears that an increase in linear frequency with a decrease in depth tends to make \bar{G}_c decrease.

Figure 10 shows the frequency correction factor for a fluid in a cylindrical tank with a conical bulkhead as a function of depth. For all three values of β , \bar{G}_c tends to increase with a decrease in depth until the fluid surface is in the conical bulkhead, at which time it starts to decrease. This can be explained as before by the fact that the linear frequency is increasing (Fig. 4) as the depth is decreasing.

Figure 11 shows the frequency correction factor for a fluid in a cylindrical tank with a truncated conical bulkhead as a function of depth. For $\beta = 30 \text{ deg}$, \bar{G}_c behaves very much like it does for a cylindrical tank, which is expected since the solution becomes invalid before it becomes small enough for the fluid to be in the bulkhead section of the tank. It is seen that \bar{G}_c for $\beta = 45 \text{ deg}$ and $\beta = 60 \text{ deg}$ tend to increase with a decrease in

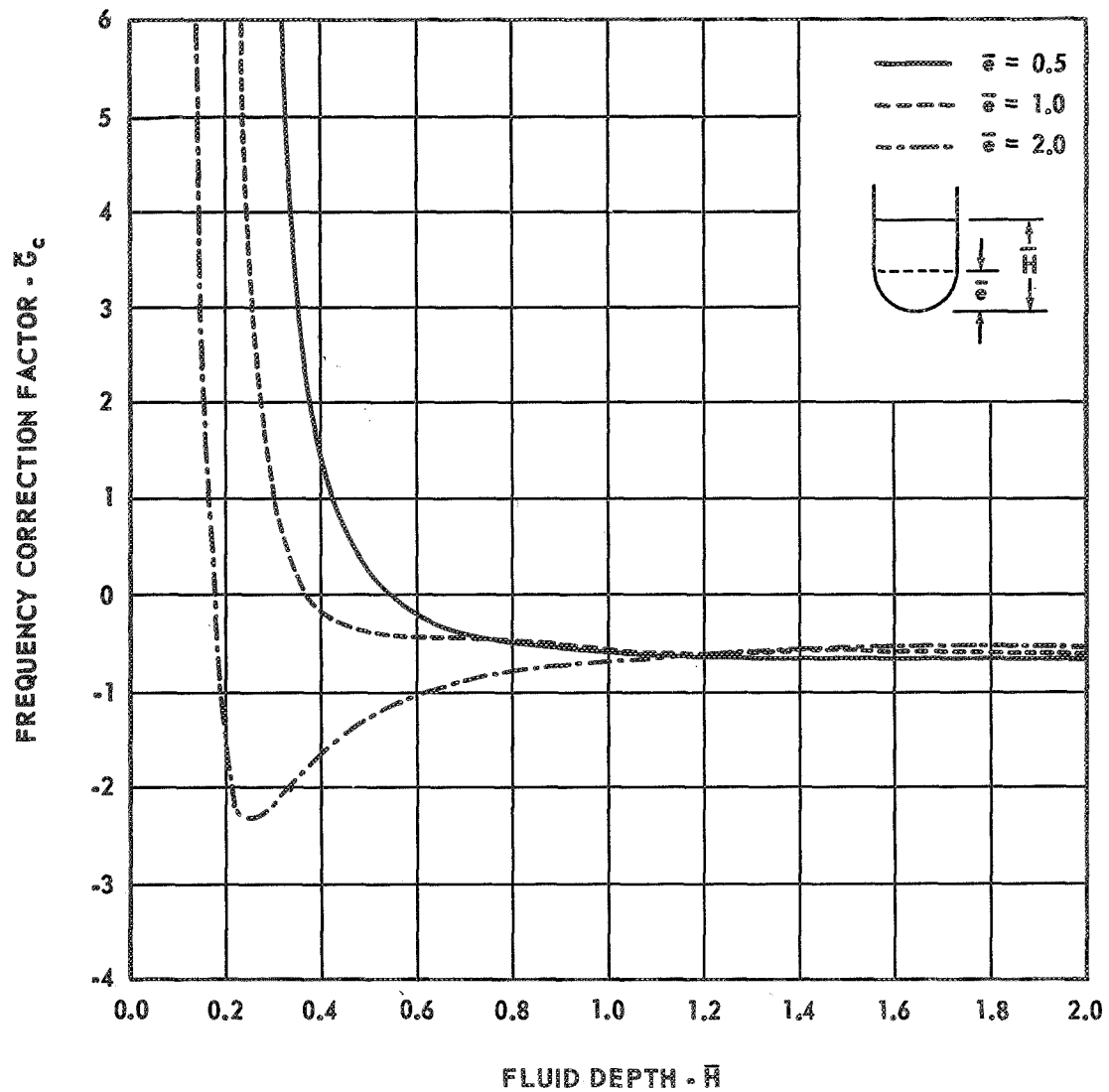


Figure 9. Frequency Correction Factor for a Fluid in a Cylindrical Tank with an Ellipsoidal Bulkhead as a Function of Depth

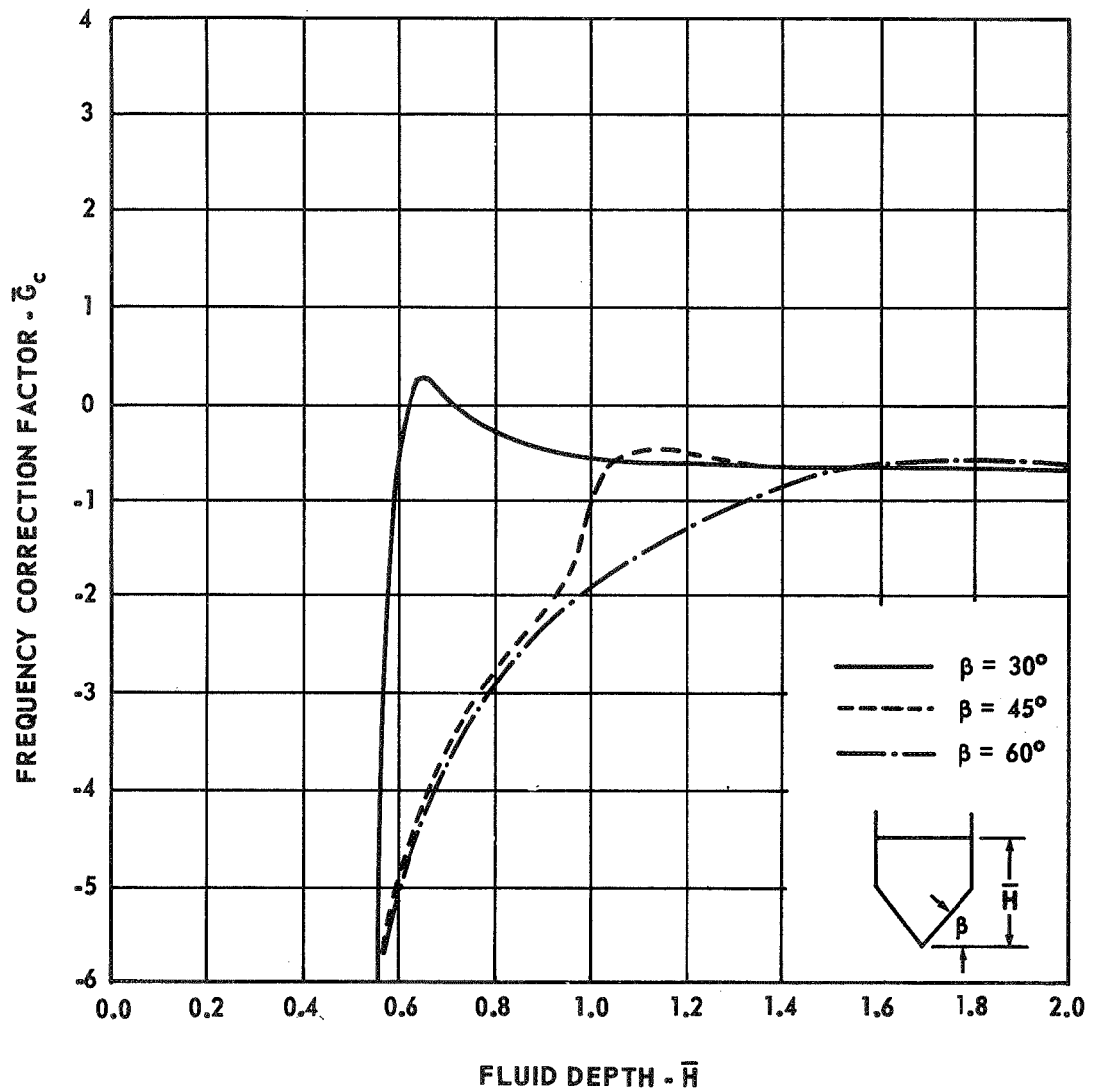


Figure 10. Frequency Correction Factor for a Fluid in a Cylindrical Tank with a Conical Bulkhead as a Function of Depth

depth until the fluid surface reaches the bulkhead, and then start to decrease until the tank bottom becomes effective.

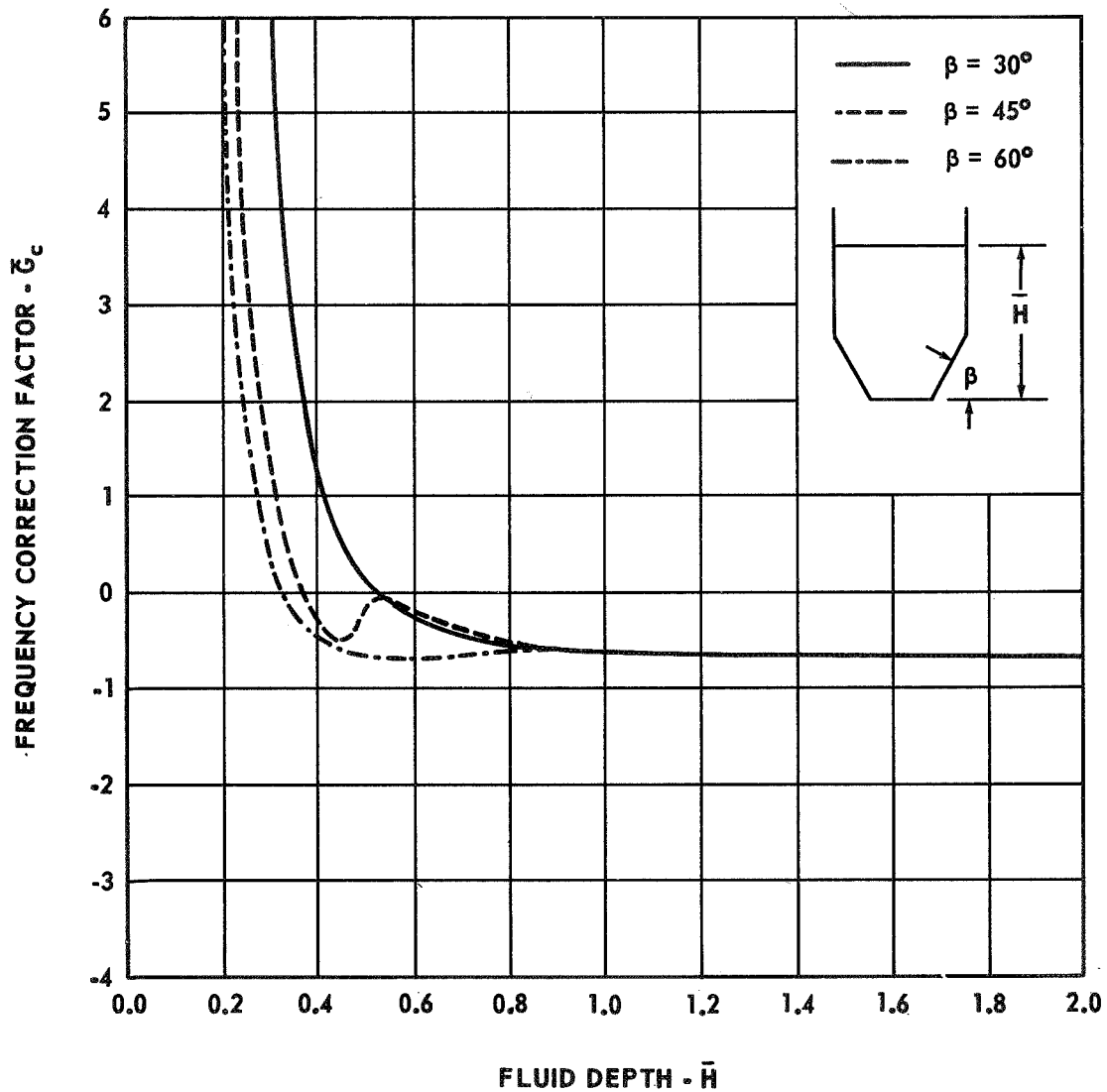


Figure 11. Frequency Correction Factor for a Fluid in a Cylindrical Tank with a Truncated Conical Bulkhead as a Function of Depth

CONCLUSIONS

A solution for finite-amplitude free oscillations of a fluid in a partially filled axisymmetric container has been presented. The formulation of the problem resulted in a nonlinear boundary value problem where the nonlinearity occurred in the boundary condition at the free surface of the fluid. The boundary condition at the container wall and the differential equation were linear.

The solution was obtained by first linearizing the free surface boundary condition and solving the resulting linear boundary value problem. Then the linear solutions, which satisfy the differential equation and the boundary condition at the container wall, are used to construct a solution that satisfies the nonlinear boundary condition at the free surface asymptotically. A solution through the third-order term was developed.

Numerical results were obtained for finite-amplitude standing waves near the first antisymmetric linear mode in three axisymmetric vessels. Results were found for a cylindrical tank with an ellipsoidal bulkhead, a cylindrical tank with a conical bulkhead, and a cylindrical tank with a truncated conical bulkhead.

The method of solving the linear problem was verified by comparing the computed linear frequency with published test results for the case of a cylindrical tank with a hemispherical bulkhead. The nonlinear solution was checked by comparing the results for a cylindrical tank with published theoretical results.

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National Aeronautics and Space Administration

Marshall Space Flight Center, Alabama 35812, August 29, 1969

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